UNIVERZITET U BEOGRADU

MATEMATIČKI FAKULTET

Asmaa M. Kanan

O DELITELJIMA NULE, INVERTIBILNOSTI I RANGU MATRICA NAD KOMUTATIVNIM POLUPRSTENIMA

doktorska disertacija

Beograd, 2013

UNIVERSITY OF BELGRADE

FACULTY OF MATHEMATICS

Asmaa M. Kanan

ON ZERO DIVISORS, INVERTIBILITY AND RANK OF MATRICES OVER COMMUTATIVE SEMIRINGS

Doctoral Dissertation

Belgrade, 2013

PODACI O MENTORU I ČLANOVIMA KOMISIJE

MENTOR:

dr Zoran Petrović

vanredni profesor, Univerzitet u Beogradu, Matematički fakultet

ČLANOVI KOMISIJE:

dr Aleksandar Lipkovski

redovni profesor, Univerzitet u Beogradu, Matematički fakultet

dr Žarko Mijajlović

redovni profesor, Univerzitet u Beogradu, Matematički fakultet

dr Branko Malešević

vanredni profesor, Univerzitet u Beogradu, Elektrotehnički fakultet

Datum odbrane:

O DELITELJIMA NULE, INVERTIBILNOSTI I RANGU MATRICA NAD KOMUTATIVNIM POLUPRSTENIMA

REZIME

Poluprsten sa nulom i jedinicom je algebarska struktura, koja generališe prsten. Naime, dok prsten u odnosu na sabiranje čini grupu, poluprsten čini samo monoid. Nedostatak oduzimanja čini ovu strukturu znatno težom za istraživanje od prstena.

Predmet izučavanja u ovoj tezi predstavljaju matrice nad komutativnim poluprstenima (sa nulom i jedinicom). Motivacija za istraživanje je sadržana u pokušaju da se ispita koje se osobine za matrice nad komutativnim prstenima mogu proširiti na matrice nad komutativnim poluprstenima, a takodje, što je tesno povezano sa ovim pitanjem, kako se svojstva modula nad prstenima prenose na polumodule nad poluprstenima.

Izdvajaju se tri tipa dobijenih rezultata.

Najpre se proširuju poznati rezultati, koji se tiču dimenzije prostora n-torki elemenata iz nekog poluprstena na drugu klasu poluprstena od do sada poznatih i ispravljaju neke greške u radu drugih autora. Ovo je pitanje u tesnoj vezi sa pitanjem invertibilnosti matrica nad poluprstenima.

Drugi tip rezultata se tiče ispitivanja delitelja nule u poluprstenu svih matrica nad komutativnim poluprstenima i to posebno za klasu inverznih poluprstena (to su poluprsteni u kojima postoji neka vrste uopshtenog inverza u odnosu na sabiranje). Zbog nepostojanja oduzimanja, ne može se koristiti determinanta, kao što je to u slučaju matrica nad komutativnim prstenima, ali, zbog činjenice da su u pitanju inverzni poluprsteni, moguće je definisati neku vrstu determinante u ovom slučaju, što omogućava formulaciju odgovarajućih rezultata u ovom slučaju. Zanimljivo je da se za klase matrica za koje se dobijaju rezultati, levi i desni delitelji nule mogu razlikovati, što nije slučaj za komutativne prstene.

Treći tip rezultata tiče se pitanja uvodjenja novog ranga za matrice nad komutativnim poluprstenima. Za ovakve matrice već postoji niz funkcija ranga, koje generališu postojeću funkciju ranga za matrice nad poljima. U ovoj tezi je predložena nova funkcija ranga, koja je bazirana na permanenti, koju je moguće definisati i za poluprstene, za razliku od determinante, a koja ima dovoljno dobra svojstva da se tako definishe rang.

Ključne reči: komutativni prsteni, poluprsteni, pozitivna determinanta, delitelji nule, invertibilnost, rang, matrice

Naučna oblast: Matematika

Uža naučna oblast: Algebra

UDK broj: 512.558(043.3)

ON ZERO DIVISORS, INVERTIBILITY AND RANK OF MATRICES OVER COMMUTATIVE SEMIRINGS

ABSTRACT

Semiring with zero and identity is an algebraic structure which generalizes a ring. Namely, while a ring under addition is a group, a semiring is only a monoid. The lack of substraction makes this structure far more difficult for investigation than a ring.

The subject of investigation in this thesis are matrices over commutative semirings (wiht zero and identity). Motivation for this study is contained in an attempt to determine which properties for matrices over commutative rings may be extended to matrices over commutative semirings, and, also, which is closely connected to this question, how can the properties of modules over rings be extended to semimodules over semirings.

One may distinguish three types of the obtained results.

First, the known results concerning dimension of spaces of n-tuples of elements from a semiring are extended to a new class of semirings from the known ones until now, and some errors from a paper by other authors are corrected. This question is closely related to the question of invertibility of matrices over semirings.

Second type of results concerns investigation of zero divisors in a semiring of all matrices over commutative semirings, in particular for a class of inverse semirings (which are those semirings for which there exists some sort of a generalized inverse with respect to addition). Because of the lack of substraction, one cannot use the determinant, as in the case of matrices over commutative semirings, but, because of the fact that the semirings in question are inverse semirings, it is possible to define some sort of determinant in this case, which allows the formulation of corresponding results in this case. It is interesting that for a class of matrices for which the results are obtained, left and right zero divisors may differ, which is not the case for commutative rings.

The third type of results is about the question of introducing a new rank for matrices over commutative semirings. For such matrices, there already exists a number of rank functions, generalizing the rank function for matrices over fields. In this thesis, a new rank function is proposed, which is based on the permanent, which is possible to define for semirings, unlike the determinant, and which has good enough properties to allow a definition of rank in such a way.

Keywords: commutative rings, semirings, positive determinant, zero divisors, invertibility, rank, matrices

Scientific field: Mathematics

Narrow scientific field: Algebra

UDC number: 512.558(043.3)

Contents

Chapter 1

Introduction

In 1894, Dedekind introduced the modernistic definition of the ideal of a ring. He took the family of all the ideals of a ring R, $Id(R)$, and defined on it the sum $(+)$, and the product $(\cdot)$. He found that the system $(Id(R), +, \cdot)$ satisfies most of the rules that the system $(R, +, \cdot)$ satisfied, but the algebraic system $(Id(R), +, \cdot)$ was not a ring, because it was not a group (under addition), it was only a commutative monoid. The system $(Id(R), +, \cdot)$ had all the properties of an important algebraic structure, which was called later a *semiring*.

In 1934, H. S. Vandiver published a paper about algebraic system consisted of a nonempty set S with two binary operations, addition $(+)$ and multiplication $(\cdot)$, such that S was a semigroup under an addition $(+)$ and a multiplication $(\cdot)$. The system $(S, +, \cdot)$ obey both distributive laws but it did not obeyed the cancelation law of addition. This system was ring-like but it was not a ring. Vandiver called this system a *semiring*. But Vandiver was later informed by R. Brauer that Dedekind introduced this concept, but it was not by the same name *semiring*. Vandiver observed in his papers, that there are semirings can be embedded into rings.

The structure of semirings was later investigated by many authors in the 1950's. These authors have investigated various aspects of the algebraic theory of semirings including embedding of semirings into richer semirings, and other details. In recent years, semirings proved to be an important tool in many areas of applied mathematics and Computer

Science. A semiring is similar to a ring, where the difference between semirings and rings is that there are no additive inverses in semirings. Therefore, all rings are semirings. For examples of semirings which are not rings are the non-negative reals $\mathbb{R}_+$, the non-negative rationals $\mathbb{Q}_+$, and the non-negative integers $\mathbb{Z}_+$ with usual addition and multiplication.

Matrix theory over various semirings is an important subject, so it has attracted the attention of many researchers working both in theoretical and applied mathematics during the last decades.

This thesis is organized as follows.

Some basic definitions and examples of semirings and related notions (linear independence, semilinear spaces, ideals, annihilators) are given in Chapter 2.

Chapter 3 is devoted to the discussion of the cardinality of bases in semilinear spaces of n-dimensional vectors over commutative zerosumfree semirings. It is not generally true that every basis has n elements. Some examples are given and the correct condition that any basis for this type of semirings has n elements is presented. Results from this chapter were published in [9].

In Chapter 4, we investigate zero divisors for matrices over commutative additively inverse semirings with zero 0 and identity 1. It is known that a matrix over a commutative ring is a zero divisor if and only if its determinant is a zero divisor. Since determinant is impossible to define for matrices over semirings, one needs to make some changes. It is possible to define a function similar to the determinant for matrices over additively inverse semirings. For matrices satisfying an additional condition on its elements this function allows us to determine whether a matrix is a zero divisor. It is interesting that in the case of commutative semirings it is not true that a square matrix is a left zero divisor if and only if it is a right zero divisor, which is true for commutative rings. These results are contained in [10].

Finally, Chapter 5 introduces a new type of rank, which we call the *permanent rank* of matrices over commutative semirings. Namely, for semirings there are a number of rank functions already defined, generalizing various aspects of the rank function for matrices over a field. The permanent of a matrix is possible to define for matrices over commutative semirings and it has good enough properties to establish some results analogous to those for matrices over rings. Some examples are given to illustrate these results.

Chapter 2

Preliminaries on semirings

In this chapter we give some basic definitions concerning semirings and matrices over semirings. For more information about these (and other) notions, we refer the reader to [3, 4, 5, 6, 7, 11, 12, 1, 13, 17].

Definition 1. *A semiring with zero 0 and identity 1, $\mathcal{L} = (L, +, \cdot, 0, 1)$ is an algebraic structure satisfying the following axioms:*

(i) $(L, +, 0)$ *is a commutative monoid;*

(ii) $(L, \cdot, 1)$ *is a monoid;*

(iii) for all $r, s, t \in L$*:* $r \cdot (s + t) = r \cdot s + r \cdot t$ *and* $(s + t) \cdot r = s \cdot r + t \cdot r$*;*

(iv) for all $r \in L$*:* $r \cdot 0 = 0 = 0 \cdot r$*;*

(v) $1 \neq 0$.

Definition 2. *A semiring $\mathcal{L}$ is commutative if $(L, \cdot, 1)$ is a commutative monoid.*

Definition 3. *A semiring $\mathcal{L}$ is called zerosumfree (or antinegative) if from $a + b = 0$, for $a, b \in L$, it follows that $a = b = 0$.*

Example 1. $([0,1],+,\cdot)$, where $[0,1]$ is the unit interval of real numbers, is a semiring, where:

$a+b=\max\{a,b\}$, $a\cdot b=\min\{a,b\}$,

or

$a+b=\min\{a,b\}$, $a\cdot b=\max\{a,b\}$,

or even

$a+b=\max\{a,b\}$, and $\cdot$ = usual product of real numbers.

Example 2. $\mathcal{L}=(\mathbb{R}\cup\{-\infty\},+,\cdot,\{-\infty\},0)$ is a semiring, where $\mathbb{R}$ is the set of all real numbers, $a+b=\max\{a,b\}$, and $a\cdot b$ stands for the usual addition in $\mathbb{R}$ for $a,b\in\mathbb{R}\cup\{-\infty\}$.

This semiring $\mathcal{L}=(\mathbb{R}\cup\{-\infty\},+,\cdot,\{-\infty\},0)$ is usually called a max-plus algebra or a schedule algebra.

Example 3. For any nonempty set X, the system $(P(X),\cup,\cap)$ consisting of the power set $P(X)$ of X under the binary operations of $\cup$ and $\cap$ is a semiring, in particular it is a commutative zerosumfree semiring, where for $A,B\in P(X), A\cup B$ may be considered as an addition on $P(X)$ and $A\cap B$ as a multiplication on $P(X)$. The system $(P(X),\cap,\cup)$ also is a commutative semiring but it is not zerosumfree semiring, because if $A\cap B=\emptyset$ does not imply $A=\emptyset$ and $B=\emptyset$.

Example 4. Let S be a nonempty set. Define on S, $+$ by $a+b=b$ and $\cdot$ by $a\cdot b=a$ for all $a,b\in S$. Then $(S,+,\cdot)$ is a semiring which is not commutative.

Example 5. The set of non-negative integers $\mathbb{Z}_+$, non-negative rational numbers $\mathbb{Q}_+$, non-negative real numbers $\mathbb{R}_+$, under the usual addition and multiplication are familiar examples of commutative zerosumfree semirings. Note that none of them is a ring.

Example 6. For each positive integer n, the set $M_n(S)$ of all $n\times n$ matrices over a semiring S is a semiring under the usual operations of matrix addition and multiplication.

Definition 4. *A semiring $(S,+,\cdot)$ is called additively inverse if $(S,+)$ is an inverse semigroup, i.e., for each $x\in S$ there is a unique $x'\in S$ such that $x=x+x'+x$ and*

$x' = x' + x + x'$. *An additively inverse commutative semiring with zero and identity is a generalization of a commutative ring with identity.*

Definition 5. *Let* $\mathcal{A} = (A, +_A, 0_A)$ *be a commutative monoid and* $\mathcal{L} = (L, +, \cdot, 0, 1)$ *is a commutative semiring. If an external multiplication* $\bullet\colon L \times A \to A$ *such that:*

(i) for all $r, s \in L$, $a \in A$: $(r \cdot s) \bullet a = r \bullet (s \bullet a)$;

(ii) or all $r \in L$, $a, b \in A$: $r \bullet (a +_A b) = r \bullet a +_A r \bullet b$;

(iii) for all $r, s \in L$, $a \in A$: $(r + s) \bullet a = r \bullet a +_A s \bullet a$;

(iv) for all $a \in A$: $1 \bullet a = a$;

(v) for all $r \in L$, $a \in A$: $0 \bullet a = r \bullet 0_A = 0_A$,

is defined, then $\mathcal{A}$ *is called a left* $\mathcal{L}$*-semimodule.*

One can analogously define the notion of right $\mathcal{L}$-semimodules.

Definition 6. *Let* $\mathcal{L} = (L, +, \cdot, 0, 1)$ *be a commutative semiring. Then a semimodule over* $\mathcal{L}$ *is called an* $\mathcal{L}$*-semilinear space. The elements of a semilinear space will be called vectors and elements of* L *scalars.*

Example 7. Let $\mathcal{L} = (L, +, \cdot, 0, 1)$ be a commutative semiring, $X \neq \emptyset$ and $A = L^X = \{f | f : X \to L\}$. Then for all $f, g \in L^X$ we define:

$$(f +_A g)(x) = f(x) + g(x),$$

$$(\text{for } x \in X, r \in L)\, (r \bullet f)(x) = r \cdot f(x).$$

We also define the zero element 0_A as the function $0_A : x \longmapsto 0$. Then $\mathcal{A} = (L^X, +_A, 0_A)$ is an $\mathcal{L}$-semilinear space.

Definition 7. *Let* $\mathcal{A}$ *be an* $\mathcal{L}$*-semilinear space. For* $\lambda_1, \ldots, \lambda_n \in L$, $a_1, \ldots, a_n \in A$, *the element* $\lambda_1 a_1 + \cdots + \lambda_n a_n \in A$ *is called a linear combination of vectors* $a_1, \ldots, a_n \in A$.

Definition 8. *Let $\mathcal{A}$ be an $\mathcal{L}$-semilinear space. Vectors $a_1, \ldots, a_n \in A$, where $n \geq 2$ are called linearly independent if none of these vectors can be expressed as a linear combination of others. Otherwise, we say that vectors $a_1, \ldots, a_n$ are linearly dependent.*

A single non-zero vector is linearly independent.

An infinite set of vectors is linearly independent if any finite subset of it is linearly independent.

Example 8. Let $A = L^n$ be the $\mathcal{L}$-semilinear space of n-dimensional vectors over a commutative semiring $\mathcal{L}$, where $\mathcal{L} = (L, +, \cdot, 0, 1) = ([0,1], +, \cdot, 0, 1)$, where $a+b = \max\{a, b\}$, $a \cdot b = \min\{a, b\}$.

(a) The following vectors are linearly independent:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad , \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

To show that, we suppose

$$(\exists c_1, c_2, \ldots, c_{n-1} \in [0,1]) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \quad \ldots \quad + c_{n-1} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \min\{c_1, 1\} \\ \vdots \\ 0 \end{pmatrix} + \quad \ldots \quad + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \min\{c_{n-1}, 1\} \end{pmatrix}$$

we note that

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ \min\{c_1, 1\} \\ \vdots \\ \min\{c_{n-1}, 1\} \end{pmatrix}$$

because $1 \neq 0$. We do the same work for other vectors $e_2, e_3, \ldots, e_n$, we find that none of them can be expressed as a linear combination of others. Hence, the vectors $e_1, e_2, \ldots, e_n$ are linearly independent.

(b) The following vectors:

$$\mathbf{f}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \ldots \quad , \mathbf{f}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}.$$

are linearly independent. To show that we suppose

$$(\exists c_1, c_2, \ldots, c_{n-1} \in [0, 1]) \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + \quad \ldots \quad + c_{n-1} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \min\{c_1, 1\} \\ 0 \\ \vdots \\ \min\{c_1, 1\} \end{pmatrix} + \quad \dots \quad + \begin{pmatrix} \min\{c_{n-1}, 1\} \\ \min\{c_{n-1}, 1\} \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ 0 \\ \vdots \\ c_1 \end{pmatrix} + \quad \dots \quad + \begin{pmatrix} c_{n-1} \\ c_{n-1} \\ \vdots \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \max\{c_1, \dots, c_{n-1}\} \\ \max\{c_2, \dots, c_{n-1}\} \\ \vdots \\ \max\{c_1, c_{n-2}\} \end{pmatrix}$$

$0 = \max\{c_1, \dots, c_{n-1}\} \Rightarrow c_1 = \cdots = c_{n-1} = 0$, but we see that $1 = \max\{c_2, \dots, c_{n-1}\}$ and this means that one of $c_2, \dots, c_{n-1}$ is equal to 1. So, this is contradiction because we found $c_1 = \cdots = c_{n-1} = 0$. So, f_1 can not be expressed as a linear combination of $f_1, \dots, f_n$. We apply the same work for other vectors $f_2, \dots, f_n$ we find that none of them can be expressed as a linear combination of others. Hence the vectors $f_1, \dots, f_n$ are linearly independent.

Example 9. Let $(S, +, \cdot) = ([0, 1], \max, \min)$ be a semiring, where $[0, 1]$ is the unit interval of real numbers, and $a + b = \max\{a, b\}$, $a \cdot b = \min\{a, b\}$. The vectors from $[0, 1]^n$:

$$a_1 = (a, 0, 0, \dots, 0),$$

$$a_2 = (0, a, 0, \dots, 0),$$

$$\dots\dots\dots\dots\dots\dots$$

$$a_n = (0, 0, 0, \dots, a),$$

where $a \in (0, 1)$, are linearly independent (it is easy to see that). But the vectors

$$a_1, \ldots, a_n, a_{n+1},$$

where $a_{n+1} = (a, a, \ldots, 0)$ are linearly dependent since $a_{n+1} = a_1 + a_2$.

We need notions of a generating set and a basis.

Definition 9. *A nonempty subset B of vectors from A is called a generating set if every vector from A is a linear combination of vectors from B.*

Definition 10. *A linearly independent generating set is called a basis.*

The notion of an invertible matrix for matrices over commutative semirings is completely analogous to the notion of invertible matrices for commutative rings.

Definition 11. *Let S be a semiring. A matrix A in $M_n(S)$ is right invertible (resp. left invertible) if $AB = I_n$ (resp. $BA = I_n$) for some $B \in M_n(S)$. In this case the matrix B is called a right inverse (resp. left inverse) of A in $M_n(S)$. The matrix A in $M_n(S)$ is called invertible if it is both right and left invertible.*

Definition 12. *We define the set $U(L)$ of (multiplicatively) invertible elements from L by:*

$$U(L) := \{a \in L | (\exists b \in L) (a \cdot b = b \cdot a = 1)\}.$$

Definition 13. *An element $r \in L$ is additively irreducible if from $r = a + b$, it follows that $r = a$ or $r = b$.*

The notion of an ideal and its annihilator is important for the study of permanent rank in the last chapter.

Definition 14. *A subset I of a semiring S is a right (resp. left) ideal of S if for $a, b \in I$ and $s \in S$, $a + b \in I$ and $ar \in I$ (resp. $ra \in I$); I is an ideal if it is both a right and a left ideal.*

Definition 15. *Let S be a commutative semiring, and let I be an ideal of S. The annihilator of an ideal I, denoted by* $\mathrm{Ann}_S(I)$*, is the set of all elements x in S such that for each y in I, $x \cdot y = 0$, i.e.,*

$$\mathrm{Ann}_S(I) \coloneqq \{x \in S \,|\, (\forall y \in I)\, x \cdot y = 0\}.$$

It is clear that annihilator is also an ideal of S.

Chapter 3

Cardinality of bases in semilinear spaces over zerosumfree semirings

3.1 Definitions and previous results

We begin this chapter with an example.

Example 10. *Let $\mathcal{L} = (L, +, \cdot, 0, 1)$ be a commutative semiring. For each $n \geq 1$, let $V_n(L) := \{(r_1, \dots, r_n)^T : r_i \in L, 1 \leq i \leq n\}$. Then, $V_n(L)$ becomes a $\mathcal{L}$-semilinear space if the operations are defined as follows:*

$$(r_1, \dots, r_n)^T + (s_1, \dots, s_n)^T = (r_1 + s_1, \dots, r_n + s_n)^T;$$

$$r \bullet (r_1, \dots, r_n)^T = (r \cdot r_1, \dots, r \cdot r_n)^T,$$

for all $r, r_i, s_j \in L$. We denote this semilinear space by $\mathcal{V}_n$ and we call it the semilinear space of n-dimensional vectors over $\mathcal{L}$.

The following question naturally arises: if a semilinear space has a basis, is it true that all bases have the same cardinality? For the discussion of the problem for a particular class of semirings (join-semirings), see [19]. In general, this is not true, as the following example shows.

Example 11. Let $\mathcal{L} = (L, \oplus, \odot, 0, 1)$ be a commutative zerosumfree semiring, where L is the non-negative integers $\mathbb{Z}_+$, and $\oplus$, $\odot$ are defined as: for a and b in L, $a \oplus b = gcd\{a, b\}$ and $a \odot b = lcm\{a, b\}$, and gcd(resp. lcm) denotes the greatest common divisor (resp. least common multiple) of a and b. We also put $a \odot 0 = 0$. Then in $\mathcal{L}$-semilinear space $\mathcal{V}_2$, the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the vectors

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

and the vectors

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

are bases of $\mathcal{V}_2$, but they have not the same number of elements — the first basis has two elements, the second basis has four elements, and the third basis has three elements. We show that these vectors form bases. We have the first basis $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. We will show that set is linearly independent, so suppose

$$\begin{aligned} (\exists m \in \mathbb{Z}_+) \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= m \odot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} m \odot 0 \\ m \odot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ m \end{pmatrix}. \end{aligned}$$

We note that

$$(\forall m \in \mathbb{Z}_+) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ m \end{pmatrix},$$

also

$$(\forall n \in \mathbb{Z}_+) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq n \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

are linearly independent. Now, we will show that vectors span $\mathcal{V}_2$: note that for any vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{V}_2$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus y \odot \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so, the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

span $\mathcal{V}_2$, that means that the set $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ is a generating set. Since that set of vectors is linearly independent and generating set, so it is a basis of $\mathcal{V}_2$. Now, will show that these vectors

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

are linearly independent: suppose that

$$\begin{aligned} (\exists m, n, p \in \mathbb{Z}_+)\begin{pmatrix} 2 \\ 0 \end{pmatrix} &= m \odot \begin{pmatrix} 3 \\ 0 \end{pmatrix} \oplus n \odot \begin{pmatrix} 0 \\ 2 \end{pmatrix} \oplus p \odot \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \begin{pmatrix} m \odot 3 \\ n \odot 2 \oplus p \odot 3 \end{pmatrix} \end{aligned}$$

this mean that $2 = lcm\{m, 3\}$, but this is impossible for any $m \in \mathbb{Z}_+$ since $3 \nmid 2$. By the same way, we find that none of other vectors can be represented by a linear combination of the others. So they are linearly independent. To show that these vectors span $\mathcal{V}_2$ note that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

So

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

span (generate) every vector in $\mathcal{V}_2$. Since that vectors are linearly independent and span $\mathcal{V}_2$, then it form a basis of $\mathcal{V}_2$. By the same way, we can find that the vectors

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

form a basis of $\mathcal{V}_2$.

Let us concentrate our attention to the case of the space of n-dimensional vectors $\mathcal{V}_n$. In [14], the authors claim that the following result is true: In $\mathcal{L}$-semilinear space $\mathcal{V}_n$, where $\mathcal{L}$ is a zerosumfree semiring, each basis has the same number of elements if and only if 1 is an additive irreducible element (see Theorem 3.1 in that paper). Alas, this result is not true. For example, in the case when $\mathcal{L} = \mathbb{Q}_+$ (by $\mathbb{Q}_+$ we denote the non-negative rational numbers with the usual addition and multiplication, as mentioned before), all bases in $\mathcal{V}_n$ have n elements, while 1 obviously is not an additive irreducible element. We show how to correct this result and we point out that everything depends on the 1-dimensional case.

3.2 Cardinality of bases in $\mathcal{V}_n$

We assume that a semiring $\mathcal{L}$ is zerosumfree. In this section we prove the main result in the following theorem.

Theorem 1. *For every $n \geq 1$, every basis in $\mathcal{V}_n$ has n elements if and only if every basis in $\mathcal{V}_1$ has 1 element.*

Of course, one only needs to prove the non-trivial part. We begin with the following lemma.

Lemma 1. *Suppose that every basis in $\mathcal{V}_1$ has one element. Then, if $a_1, \ldots, a_m \in L$ are such that $a_1 + \cdots + a_m = 1$, it follows that at least one of a_i is invertible.*

Proof. We prove this by induction on m.

The claim is true for $m = 1$. Suppose that it is true for $m (\geq 1)$; we prove it for $m + 1$.

So, suppose that

$$a_1 + \cdots + a_{m+1} = 1. \tag{3.2.1}$$

Then $\{a_1, \ldots, a_{m+1}\}$ is a generating set for $\mathcal{V}_1$. Since every basis for $\mathcal{V}_1$ has 1 element, this set is not a basis. So, at least one of the elements is a linear combination of others. Suppose, for example, that a_{m+1} is a linear combination of others, so

$$a_{m+1} = \lambda_1 a_1 + \cdots + \lambda_m a_m, \tag{3.2.2}$$

for some $\lambda_1, \ldots, \lambda_m \in L$. Substituting this expression for a_{m+1} into equation (3.2.1) we get

$$(1 + \lambda_1) a_1 + \cdots + (1 + \lambda_m) a_m = 1. \tag{3.2.3}$$

By induction hypothesis, at least one of $(1+\lambda_i)a_i$ is invertible. It follows that a_i is invertible also, and we are done. $\square$

Proof of Theorem 1. We assume that every basis of $\mathcal{V}_1$ has 1 element. Suppose that $A_1, \ldots, A_m$ is a basis for $\mathcal{V}_n$ with

$$A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix},$$

for $a_{ij} \in L$.

We have the canonical basis for $\mathcal{V}_n$: $\mathbf{e}_1, \ldots, \mathbf{e}_n$ given by

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots \quad , \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Since $A_1, \ldots, A_m$ is also a basis, the vectors from the canonical basis may be expressed as linear combinations of $A_1, \ldots, A_m$. As in ordinary linear algebra, this may be expressed as a product of matrices:

$$A \cdot B = I_n,$$

where vectors $A_1, \ldots, A_m$ are columns of the matrix A, and I_n is the identity matrix of order n whose columns are exactly vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

From this equation, by looking at the first column of the product, we get

$$\begin{aligned} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} &= 1 \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &\ \ldots\ . \\ a_{n1}b_{11} + a_{n2}b_{21} + \cdots + a_{nm}b_{m1} &= 0. \end{aligned}$$

From Lemma 1, it follows that at least one of $a_{1k}b_{k1}$ is invertible. We may assume without loss of generality that it is $a_{11}b_{11}$ that is invertible. Since $\mathcal{L}$ is zerosumfree, we conclude that

$$a_{21}b_{11} = \cdots = a_{n1}b_{11} = 0.$$

Since $b_{11} \in U(L)$, it follows that $a_{21} = \cdots = a_{n1} = 0$. So, $A_1 = a_{11}\mathbf{e}_1$.

We proceed to the second column of the product of matrices A and B.

$$
\begin{array}{rcl}
a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1m}b_{m2} & = & 0 \\
a_{22}b_{22} + \cdots + a_{2m}b_{m2} & = & 1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & . \\
a_{n2}b_{22} + \cdots + a_{nm}b_{m2} & = & 0.
\end{array}
$$

As before, from Lemma 1, it follows that $a_{2k}b_{k2} \in U(L)$ for some k. For simplicity of notation, let us assume that $a_{22}b_{22} \in U(L)$. So, as in the previous case, we get that $A_2 = a_{22}\mathbf{e}_2$.

We perform this process for $A_1, \ldots, A_l$, where $l = \min\{m, n\}$ and we get that, for some injective function $\pi : \{1, \ldots, l\} \to \{1, \ldots, n\}$, $A_k = a_{\pi(k)\pi(k)}\mathbf{e}_{\pi(k)}$ for $1 \leq k \leq l$ and $a_{\pi(k)\pi(k)} \in U(L)$.

1. If $m < n$, then $l = m$ and let s be an element of $\{1, \ldots, n\}$ which is not in the image of π (in this case π cannot be onto). It is clear that $\mathbf{e}_s$ is not a linear combination of $A_1, \ldots, A_m$, so these vectors cannot form a basis. Therefore, $m \geq n$.

2. If $m > n$, we obtain that $A_1, \ldots, A_n$ already form a basis. Therefore, we must have $m = n$. □

3.3 Additional remarks

We have proved that is enough to assume that every basis in $\mathcal{V}_1$ has 1 element in order to conclude that for any n every basis in $\mathcal{V}_n$ has n elements. One may express the fact that every basis in $\mathcal{V}_1$ has 1 element in various equivalent ways and one of them is the following: every basis in $\mathcal{V}_1$ has 1 element if and only if from $a + b = 1$, where $a, b \in L$ it follows that $a \in U(L)$, or $b \in U(L)$. Instead of using this condition as a necessary and sufficient condition that the cardinality of every basis of $\mathcal{V}_n$ is n, we have decided to present the result in the form of Theorem 1 in order to emphasize that everything depends on $\mathcal{V}_1$.

Chapter 4

Zero divisors for matrices over commutative semirings

It is known that a square matrix A over a commutative ring R with identity is a left (right) zero divisor in $M_n(R)$ if and only if the determinant of A is a zero divisor in R (see [2]). Additively inverse commutative semirings with zero 0 and identity 1 are a generalization of commutative rings with identity. In this chapter, we present some results for matrices over this type of semirings which generalize the above result for matrices over commutative rings. For the results concerning invertibility of matrices over semirings, we refer the reader to [13, 15, 16, 18].

4.1 Preliminaries

In this section, we collect only the necessary notions for the presentation of the main result in the last section.

Definition 16. *An element x of a semiring $(S, +, \cdot)$ with zero 0 (identity 1) is said to be additively (multiplicatively) invertible if $x + y = y + x = 0$ ($x \cdot y = y \cdot x = 1$) for some unique $y \in S$.*

Definition 17. *A square matrix A over a commutative semiring S is called a left zero divisor in $M_n(S)$ if $AB = O$ for some nonzero matrix $B \in M_n(S)$. Similarly, A is called a right zero divisor in $M_n(S)$ if $CA = O$ for some nonzero matrix $C \in M_n(S)$.*

Definition 18. *[15] Let $\mathcal{S}_n$ be the symmetric group of degree $n \geq 2$, $\mathcal{A}_n$ the alternating group of degree n, and $\mathcal{B}_n = \mathcal{S}_n \backslash \mathcal{A}_n$, that is,*

$$\mathcal{A}_n = \{\sigma \in \mathcal{S}_n : \sigma \textit{ is an even permutation}\},$$

$$\mathcal{B}_n = \{\sigma \in \mathcal{S}_n : \sigma \textit{ is an odd permutation}\}.$$

For $A \in M_n(S)$, the positive determinant and the negative determinant of A are defined respectively, as follows:

$$\det{}^+ A = \sum_{\sigma \in \mathcal{A}_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right),$$

$$\det{}^- A = \sum_{\sigma \in \mathcal{B}_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right).$$

We can see that $\mathcal{A}_n = \{\sigma^{-1} : \sigma \in \mathcal{A}_n\}$ and $\mathcal{B}_n = \{\sigma^{-1} : \sigma \in \mathcal{B}_n\}$, $\det^+ I_n = 1$ and $\det^- I_n = 0$ and for $A \in M_n(S)$, $\det^+ A^t = \det^+ A$, $\det^- A^t = \det^- A$(see [6]).

Proofs of the following lemma may be found in [15].

Lemma 2. *For distinct $i, j \in \{1, \ldots, n\}, \sigma \longmapsto (\sigma(i), \sigma(j))\sigma$ is a bijection from $\mathcal{A}_n$ onto $\mathcal{B}_n$.*

We also need the following propositions.

Proposition 1. *If $(S, +, \cdot)$ is an additively inverse semiring, then for all $x, y \in S$,*

(i) $(x')' = x$,

(ii) $(x + y)' = y' + x'$,

(iii) $(xy)' = x'y = xy'$,

(iv) $x'y' = xy$.

Proposition 2. *If* $(S, +, \cdot)$ *is an additively inverse semiring with zero 0 and* $x, y \in S$ *are such that* $x + y = 0$, *then* $y = x'$.

Proposition 3. *(i)* $(x')' = x$,

(ii) $(x + y)' = y' + x'$,

(iii) $(xy)' = x'y = xy'$,

(iv) $x'y' = xy$.

Proposition 4. *If* $(S, +, \cdot)$ *is an additively inverse semiring with zero 0 and* $x, y \in S$ *are such that* $x + y = 0$, *then* $y = x'$.

It is known that a commutative semiring S may be embedded into a ring if and only if it satisfies the additive cancellation law: if $a + x = b + x$, it follows that $x = y$.

Example 12. There are additively inverse semirings which cannot be embedded into a ring. For example, let $(S, +, \cdot) = ([0, 1], \oplus, \odot)$, where $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $x \oplus y := \max\{x, y\}$ and $x \odot y := \min\{x, y\}$. We know that this is an inverse semiring (see, e.g. [15]). However, since $1/2 + 1 = 1 = 1/3 + 1$, and $1/2 \neq 1/3$, this semiring does not satisfy additive cancellation law, so it cannot be embedded into a ring.

4.2 Auxiliary results

In the following, all semirings will be inverse semirings with zero 0 and identity 1. In particular, in $M_n(S)$ there is the identity matrix I_n, all of whose diagonal elements equal to 1, and all non-diagonal elements equal to 0. It is clear that $A \cdot I_n = I_n \cdot A = A$ for every $A \in M_n(S)$.

Let S be an additively semiring, $a \in S$ and $n \geq 0$, a non-negative integer. We define $a^{(n)}$ as follows:

$$a^{(n)} := \begin{cases} a, & \text{if } n \text{ is even} \\ a', & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to check that $(a^{(n)})^{(m)} = a^{(n+m)}$ and that $a^{(n)}a^{(m)} = a^{(n+m)}$.

Using this notation, we define the determinant for matrices over additively inverse semirings.

Definition 19. *Let S be an additively inverse semiring and $A \in M_n(S)$. Then, we define $\widetilde{\det}(A)$ as follows:*

$$\widetilde{\det}(A) := \det^+(A) + (\det^-(A))'.$$

Note that, if we put $\operatorname{sgn}(\sigma) = 0$, for even pertumations, and $\operatorname{sgn}(\sigma) = 1$, for odd permutations, we have that

$$\widetilde{\det}(A) = \sum_{\sigma \in \mathcal{S}_n} \left(\prod_{i=1}^{n} a_{i\sigma(i)} \right)^{\operatorname{sgn}(\sigma)},$$

which is completely analogous to the usual expansion of the determinant.

We can also define the adjoint matrix of a given matrix in $M_n(S)$ as follows.

Definition 20. *Let S be an additively inverse semiring and $A \in M_n(S)$. For $i, j \in \{1, \ldots, n\}$, let $\mathcal{M}_{ij}(A) \in M_{n-1}(S)$ be a matrix obtained from the matrix A by deleting the ith row and jth column from this matrix. Then $\widetilde{\operatorname{adj}}(A) \in M_n(S)$ is defined as:*

$$\widetilde{\operatorname{adj}}(A)_{ij} = (\widetilde{\det}(\mathcal{M}_{ji}(A)))^{(i+j)}.$$

This definition is, as in the case of $\widetilde{\det}(A)$, completely analogous to the usual definition of the adjoint matrix. It is easy to see that the Laplace expansion with respect to any row holds in our case for $\widetilde{\det}$ (the proof is completely analogous to the usual case, so we omit it).

Theorem 2. *If S is an additively inverse semiring, $A \in M_n(S)$ and $i \in \{1, \ldots, n\}$, one has:*

$$\widetilde{\det}(A) = \sum_{j=1}^{n} a_{ij} (\widetilde{\det}(\mathcal{M}_{ij}(A)))^{(i+j)}.$$

Example 13. Let us assume that ab is additively invertible and let $A \in M_2(S)$ be the matrix

$$A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

Then $\widetilde{\det}(A) = ab + ba' = ab + (ab)' = 0$.

Keeping this example in mind, one should not be surprised that the following theorem holds.

Theorem 3. *Let S be an additively inverse semiring and $A \in M_n(S)$ be a square matrix such that for all $i, j, k,\ j \neq k$ the elements $a_{ij}a_{ik}$ are additively invertible. If, in addition to that, A has two equal rows, then $\widetilde{\det}(A) = 0$.*

Proof. In addition to the simplest Laplace expansion mentioned in 2, the more general expansion with respect to any k rows also holds. We use this expansion with respect to the two equal rows, and since any 2×2 submatrix formed from these two rows has zero determinant (see the previous example), we conclude that all the terms in the expansion are zero, therefore $\widetilde{\det}(A) = 0$.

Alternatively, if the equal rows are rth and sth row, we can first expand $\widetilde{\det}(A)$ with respect to the rth row, and then expand all the terms with respect to the sth row. Then all the terms in this final expansion can be collected into pairs which cancel each other, since these two rows are equal and the corresponding elements are additively invertible. □

Example 14. It is not necessarily the case that the determinant of a matrix with two equal rows is zero. For example, let $(S, +, \cdot) = ([0, 1], \oplus, \odot)$, the semiring from Example 1. Then for

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

we have $\widetilde{\det}(A) = 1 \cdot 1 + 1 \cdot 1' = 1 + 1 = 1$.

The following theorem is vital for our main results.

Theorem 4. *Let S be an additively inverse semiring, $A \in M_n(S)$ is such that $a_{ij}a_{ik}$ are additively invertible for all i, j, k, such that $j \neq k$. Then*

$$A \cdot \widetilde{\mathrm{adj}}(A) = \widetilde{\det}(A) I_n,$$

where $I_n \in M_n(S)$ is the identity matrix.

Proof. The proof is standard. The (i,k)th component of the product is:

$$(A\cdot\widetilde{\mathrm{adj}}(A))_{ik}=\sum_{j=1}^{n}a_{ij}\widetilde{\mathrm{adj}}(A))_{jk}=\sum_{j=1}^{n}a_{ij}(\widetilde{\det}(\mathcal{M}_{kj}(A)))^{(k+j)}.$$

If $i=k$, we have:

$$(A\cdot\widetilde{\mathrm{adj}}(A))_{ii}=\sum_{j=1}^{n}a_{ij}\widetilde{\mathrm{adj}}(A))_{ji}=\sum_{j=1}^{n}a_{ij}(\widetilde{\det}(\mathcal{M}_{ij}(A)))^{(i+j)}=\widetilde{\det}(A).$$

If $i\neq k$ then we also have an expansion of the determinant of a matrix, but in this case this matrix has equal ith and kth row. So, from Theorem 3, we conclude that this sum is equal to zero. □

Remark 1. *One can check that the equality $A\cdot\widetilde{\mathrm{adj}}(A)=\widetilde{\det}(A)I_n$ need not be true for all matrices $A\in M_n(S)$. One can check the same matrix as in Example 3.*

4.3 Main results

In this section we prove the main results.

Theorem 5. *Let S be an additively inverse semiring with zero 0 and identity 1 and $A\in M_n(S)$ is such that $a_{ij}a_{ik}$ is additively invertible for all $i,j,k,\ j\neq k$. If A is a right zero divisor, then $\widetilde{\det}(A)$ is a zero divisor.*

Proof. Since A is a right zero divisor, there exists a non-zero matrix $B\in M_n(S)$ such that $B\cdot A=O$. If we multiply this equality on the right by $\widetilde{\mathrm{adj}}(A)$, we get

$$B\cdot A\cdot\widetilde{\mathrm{adj}}(A)=O,$$

and taking into account results from Theorem 4, we get

$$B\widetilde{\det}(A)=O.$$

Since $B\neq O$, there is a component $b_{ij}\neq 0$, such that $b_{ij}\widetilde{\det}(A)=0$, so $\widetilde{\det}(A)$ is a zero divisor. □

Theorem 6. *Let S be an additively inverse semiring with zero 0 and identity 1 and $A \in M_n(S)$ is such that $a_{ij}a_{ik}$ is additively invertible for all $i, j, k,\ j \neq k$. If $\widetilde{\det}(A)$ is a zero divisor, then A is a left zero divisor.*

Proof. Since $\widetilde{\det}(A)$ is a zero divisor, there exists $x \in S$ such that $\widetilde{\det}(A) \cdot x = 0$. If $a_{ij} \cdot x = 0$ for all i, j, then $A \cdot (xI_n) = O$ and we are done.

So, let us assume that there exist i, j such that $a_{ij} \cdot x \neq 0$. Since $\widetilde{\det}(A) \cdot x = 0$, there must exist maximal r such that $1 \leq r \leq n-1$ and x annihilates all determinants of submatrices of A of order $r+1$, while there exists a submatrix C of order r of A such that $\widetilde{\det}(C) \cdot x \neq 0$. We may assume without loss of generality (and in order to simplify notation) that it is the submatrix formed by the first r rows and columns of matrix A. Let us denote by B, the submatrix of A formed by the first $r+1$ rows and columns of A. We claim that the following product is equal to zero.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} & a_{1,r+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2r} & a_{2,r+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} & a_{r,r+1} & \cdots & a_{rn} \\ a_{r+1,1} & a_{r+1,2} & \cdots & a_{r+1,r} & a_{r+1,r+1} & \cdots & a_{r+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} & a_{n,r+1} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} (\mathcal{M}_{r+1,1}(B))^{(r+2)}x \\ (\mathcal{M}_{r+1,2}(B))^{(r+3)}x \\ \vdots \\ (\mathcal{M}_{r+1,r+1}(B))^{(2r+2)}x \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

If $1 \leq i \leq r$ we have

$$\sum_{j=1}^{r+1} a_{ij}(\mathcal{M}_{r+1,j}(B))^{(r+1+j)} = 0,$$

since this is just the determinant of a matrix of order $r+1$ having the same ith and $r+1$st row (we replace $r+1$st row of matrix B with its ith row).

For $i = r+1$, we have

$$\sum_{j=1}^{r+1} a_{r+1,j}(\mathcal{M}_{r+1,j}(B))^{(r+1+j)}x = \widetilde{\det}(B)x = 0,$$

since x annihilates all determinants of submatrices of A of order larger than r. The same conclusion holds for $i > r+1$, since the corresponding sum is just determinant (or determinant') of a submatrix of order larger than r of matrix A.

On the hand, $(\mathcal{M}_{r+1,r+1}(B))^{(2r+2)}x = \widetilde{\det}(C)x \neq 0$ by assumption, so this column is not equal to zero. If we add $n-1$ zero columns to this one, we obtain a matrix $D \neq O$, such that $A \cdot D \neq O$ and we conclude that A is a left zero divisor. $\square$

Corollary 1. *Let $A \in M_n(S)$ be a matrix with entries in an additively inverse semiring S, such that $a_{ij}a_{ik}$ is additively invertible for all i, j, k, $i \neq j$. Then, if A is a right zero divisor, then A is a left zero divisor.*

Proof. The proof follows directly from previous theorems. Namely, if A is a right zero divisor, then by Theorem 5, $\widetilde{\det}(A)$ is a zero divisor, so, by Theorem 6 A, is a left zero divisor. $\square$

Remark 2. The set of left zero divisors in $M_n(S)$ may differ from the set of right zero divisors, even if the conditions concerning its components as in previous theorems hold. This is shown in the following example.

Example 15. Let $(S, +, \cdot) = ([0,1], \oplus, \odot)$, the semiring from Example 1. The matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is a left zero divisor, but it is not a right zero divisor. Namely,

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so A is a left zero divisor. Let us show that A is not a right zero divisor. Suppose that $B \in M_n(S)$ is such that $B \cdot A = O$. So, if

$$B = \begin{pmatrix} x & y \\ z & t \end{pmatrix},$$

we get

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $\max\{x, y\} = x \oplus y = 0$ and $\max\{z, t\} = z \oplus t = 0$. We conclude that $x = y = z = t = 0$. So, $B = O$ and A is not a right zero divisor.

Chapter 5

The Permanent Rank of Matrices over Commutative Semirings

5.1 Rank functions

There are many essentially different rank functions for matrices over semirings. All the rank functions coincide for matrices over fields, but they are essentially different for matrices over semirings. Also, these ranks do not coincide with the usual rank function even if a semiring S is a field. We collect well-known concepts of them. For more detailed exposition, see [8].

Definition 21. *The factor rank of the matrix $A \in M_{m\times n}(S)$, $A \neq O$ is the smallest positive integer k such that there exist matrices $B \in M_{m\times k}(S)$, and $C \in M_{k\times n}(S)$ such that $A = BC$. The factor rank of A is denoted by $rank(A)$. The factor rank of the zero matrix O is assumed to be 0.*

Definition 22. *The term rank of the matrix $A \in M_{m\times n}(S)$ is the minimum number of lines (rows or columns) needed to include all nonzero elements of A. The term rank of A is denoted by $t(A)$.*

Definition 23. *The zero-term rank of $A \in M_{m\times n}(S)$ is the minimum number of lines (rows* *or columns) needed to include all zero elements of A. The zero-term rank of A is denoted by $z(A)$.*

Definition 24. *The row (resp. column) rank of $A \in M_{m\times n}(S)$ is the dimension of the linear span of the rows (resp. columns) of A. The row (resp. column) rank of A is denoted by $r(A)$ (resp. $c(A)$).*

Definition 25. *The spanning row (resp. column) rank of $A \in M_{m\times n}(S)$ is the minimum number of rows (resp. columns) that span all rows (resp. columns) of A. The spanning row (resp. spanning column) rank of A is denoted by $sr(A)$ (resp. $sc(A)$).*

Definition 26. *The matrix $A \in M_{m\times n}(S)$ has maximal raw (resp. column) rank k if it has k linearly independent rows (resp. columns) and any $(k+1)$ rows (resp. columns) are linearly dependent. The maximal row (resp. column) rank of A is denoted by $mr(A)$ (resp. $mc(A)$).*

Now, we can define the usual rank of a matrix A over a field F as following:

Definition 27. *The rank of a matrix $A \in M_{m\times n}(F)$ is the number of linearly independent rows or columns of A.*

Example 16. Let $(S, +, \cdot) = (\mathbb{Z}_+, +, \cdot)$, and

$$A = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} \in M_3(S)$$

Then, the term rank of A is equal to two ($t(A) = 2$), because the minimum number of lines needed to include all nonzero elements of A is equal to two.

The zero-term rank of A is equal to three (z(A)=3), because the minimum number of lines needed to include all zero elements of A is equal to three.

Now, to find the factor rank of A, we need to work a little. The factor rank of A is not equal to one, because we can not write A as product of two matrices, $B \in M_{3\times 1}(\mathbb{Z}_+)$ and $C \in M_{1\times 3}(\mathbb{Z}_+)$. We can explane that in the following steps.

Suppose we can write A as product of that matrices $B \in M_{3\times1}(\mathbb{Z}_+)$ and $C \in M_{1\times3}(\mathbb{Z}_+)$ as follows:

$$\begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} a & b & c \end{pmatrix}$$

$$= \begin{pmatrix} ax & bx & cx \\ ay & by & cy \\ az & bz & cz \end{pmatrix}.$$

This corresponds to the system of equations:

$$ax = 3 \tag{5.1.1}$$

$$bx = 4 \tag{5.1.2}$$

$$cx = 0 \tag{5.1.3}$$

$$ay = 0 \tag{5.1.4}$$

$$by = 0 \tag{5.1.5}$$

$$cy = 3 \tag{5.1.6}$$

$$az = 0 \tag{5.1.7}$$

$$bz = 0 \tag{5.1.8}$$

$$cz = 3 \tag{5.1.9}$$

Since $cx = 0$, this implies $c = 0$ or $x = 0$. But we can not take $c = 0$ because that does not satisfies the equations (5.1.6) and (5.1.9). Also, we can not take $x = 0$, because that does not satisfies the equations (5.1.1) and (5.1.2). We conclude that there is no solution to these equations, hence we cannot write A as product of two matrices, $B \in M_{3\times1}(\mathbb{Z}_+)$ and $C \in M_{1\times3}(\mathbb{Z}_+)$. So, the factor rank of A is not equal to one.

Now, is the factor rank of A is equal to two or not? Let us try to find two matrices $B \in M_{3\times2}(\mathbb{Z}_+)$ and $C \in M_{2\times3}(\mathbb{Z}_+)$ such that $A = BC$.

$$\begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{pmatrix} \cdot \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1x_1 + b_1x_2 & a_2x_1 + b_2x_2 & a_3x_1 + b_3x_2 \\ a_1y_1 + b_1y_2 & a_2y_1 + b_2y_2 & a_3y_1 + b_3y_2 \\ a_1z_1 + b_1z_2 & a_2z_1 + b_2z_2 & a_3z_1 + b_3z_2 \end{pmatrix}.$$

This corresponds to the system of equations:

$$a_1x_1 + b_1x_2 = 3 \tag{5.1.10}$$

$$a_2x_1 + b_2x_2 = 4 \tag{5.1.11}$$

$$a_3x_1 + b_3x_2 = 0 \tag{5.1.12}$$

$$a_1y_1 + b_1y_2 = 0 \tag{5.1.13}$$

$$a_2y_1 + b_2y_2 = 0 \tag{5.1.14}$$

$$a_3y_1 + b_3y_2 = 3 \tag{5.1.15}$$

$$a_1z_1 + b_1z_2 = 0 \tag{5.1.16}$$

$$a_2z_1 + b_2z_2 = 0 \tag{5.1.17}$$

$$a_3z_1 + b_3z_2 = 3 \tag{5.1.18}$$

From (5.1.13), we have $a_1y_1 = 0$ which implies $a_1 = 0$ or $y_1 = 0$. Suppose that $a_1 = 0$.

By substitution $a_1 = 0$ in (5.1.10) we get $b_1x_2 = 3$ which implies $b_1 \neq 0$ and $x_2 \neq 0$. From (5.1.16), we have $b_1z_2 = 0$ which implies $z_2 = 0$. From (5.1.17) we have $a_2z_1 = 0$, this implies $a_2 = 0$ or $z_1 = 0$. Suppose that $a_2 = 0$.

By substitution $a_2 = 0$ in (5.1.11) we get $b_2x_2 = 4$, this means that $b_2 \neq 0$ and $x_2 \neq 0$. From (5.1.18) we have $a_3z_1 = 3$, this means that $a_3 \neq 0$ and $z_1 \neq 0$. From (5.1.12) we have $a_3x_1 = 0$ and $b_3x_2 = 0$, since $x_2 \neq 0$ so $b_3 = 0$, and since $a_3 \neq 0$ so $x_1 = 0$. From (5.1.10) we have $b_1x_2 = 3$ this means $b_1 = 1$ and $x_2 = 3$, or $b_1 = 3$ and $x_2 = 1$, but from (5.1.11) we have $b_2x_2 = 4$, if $b_1 = 1$ and $x_2 = 3$ in (5.1.10) then this does not satisfy

(5.1.11), because $b_2 x_2 = 4$, so it must to be $b_1 = 3$ and $x_2 = 1$, hence $b_2 = 4$ in (5.1.11). From equations (5.1.15) and (5.1.18), we have $a_3 y_1 = 3$ and $a_3 z_1 = 3$ respectively, these imply $a_3 = 3, y_1 = z_1 = 1$ or $a_3 = 1, y_1 = z_1 = 3$. Suppose that $a_3 = 1, y_1 = z_1 = 3$. Now, we have

$$A = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \\ 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 3 & 4 & 0 \end{pmatrix}.$$

Hence, the factor rank of A is equal to two, $rank(A) = 2$.

Let us determine the column rank of A ($c(A)$).

Is $c(A) = 2$?

$$\mathscr{L}\left\{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}\right\} = \left\{m \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + n \cdot \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + p \cdot \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} : m, n, p \in \mathbb{Z}\right\}$$

$$= \left\{\begin{pmatrix} 3m + 4n \\ 3p \\ 3p \end{pmatrix} : m, n, p \in \mathbb{Z}\right\}.$$

Suppose that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

form a generating set for this space, for some $a, b, c, d, e, f \in \mathbb{Z}_+$. So,

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = r \begin{pmatrix} a \\ b \\ c \end{pmatrix} + s \begin{pmatrix} d \\ e \\ f \end{pmatrix}$$

for some $r, s \in \mathbb{Z}_+$. It follows that

$$ra + sd = 3 \qquad (5.1.19)$$

$$rb + se = 0 \qquad (5.1.20)$$

$$rc + sf = 0 \qquad (5.1.21)$$

So, $rb = se = rc = sf = 0$.

1. If $r \neq 0$ and $s \neq 0$, we get $b = e = c = f = 0$. So, our vectors would be

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}.$$

It is clear that

$$\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \notin \mathscr{L}\left\{\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}\right\},$$

so this case is impossible.

2. If $r = 0$, $s \neq 0$, we get that $3 = sd$ and $e = f = 0$. Our vectors would be

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix},$$

where $sd = 3$ for some $s \in \mathbb{Z}_+$. So, $d \in \{1, 3\}$. If $d = 1$, we would get

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \left\{\begin{pmatrix} 3m + 4n \\ 3p \\ 3p \end{pmatrix} : m, n, p \in \mathbb{Z}\right\}.$$

So, $1 = 3m + 4n$ for some $m, n \in \mathbb{Z}_+$. This is clearly impossible and we conclude that $d = 3$. So the generating vectors would be

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix},$$

So,

$$\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = t\begin{pmatrix} a \\ b \\ c \end{pmatrix} + u\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix},$$

for some $s, t, \in \mathbb{Z}_+$. We get

$$\begin{aligned} ta + 3u &= 0 \qquad (5.1.22)\\ tb &= 3 \qquad (5.1.23)\\ tc &= 3. \qquad (5.1.24)\end{aligned}$$

From (5.1.23) it follows that $t \neq 0$ and from that and (5.1.22) it follows that $a = 0$. From (5.1.23) and (5.1.24) it follows that $b = c$. Our generating vectors would be

$$\begin{pmatrix} 0 \\ b \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

So,

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} = v \begin{pmatrix} 0 \\ b \\ b \end{pmatrix} + w \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

for some $v, w \in \mathbb{Z}_+$. It follows that $4 = 3w$, so $3 \mid 4$ which is not true. So, we get a contradiction.

3. The case $r \neq 0$, $s = 0$ is analogous to this case — it is also impossible.

4. $r = 0$, $s = 0$ is also impossible, since $3 \neq 0$.

We conclude that $c(A) = 3$.

Similar discussion shows that the row rank of A, $r(A)$ is equal to 3.

Let us determine the spanning column rank of A ($sc(A)$). Is $sc(A) = 1$? We note that

$$(\forall m \in \mathbb{Z}_+)\, m \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$$

because

$$(\forall m \in \mathbb{Z}_+)\, 3m \neq 4,$$

so, $sc(A) \neq 1$.

Is $sc(A) = 2$? We note that

$$(\forall m, n \in \mathbb{Z}_+)\, m \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + n \cdot \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

If

$$(\exists m, n \in \mathbb{Z}_+)\, m \cdot \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + n \cdot \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix},$$

we get $3 \mid 4$. The other case is similar to this; so, $sc(A) \neq 2$. Hence, the spanning column rank of A is equal to 3, i.e., $sc(A) = 3$.

In the same way we can find the spanning row rank of A ($sr(A)$) and $sr(A) = 2$. Namely,

$$(\forall m \in \mathbb{Z}_+)\, m \cdot \begin{pmatrix} 3 & 4 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 3 \end{pmatrix}$$

so, $sr(A) \neq 1$. So, it is clear that $sr(A) = 2$.

The maximal column rank of A ($mc(A)$). We can check as before that all the columns are linearly independent (we have practically done that in the discussion of the column rank), so $mc(A) = 3$. Similarly, for maximal row rank we have $mr(A) = 2$.

Example 17. In this example we will work over the ring $R = \mathbb{Z}$, i.e., $(R, +, \cdot) = (\mathbb{Z}, +, \cdot)$, on the same last matrix, to find all the ranks. We have

$$A = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix}.$$

We find the same results that we get in the semiring $\mathbb{Z}_+$ about: $t(A)$, $z(A)$, $rank(A)$, $r(A)$, $sc(A)$, $sr(A)$, $mc(A)$ and $mr(A)$, where $t(A) = 2$, $z(A) = 3$, $rank(A) = 2$, $r(A) = 2$, $sr(A) = 2$, $mc(A) = 2$, and $mr(A) = 2$. But $c(A) \neq 3$ as in the semiring $\mathbb{Z}_+$. Note that

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

and we get that the basis for the column space is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\},$$

hence, $c(A) = 2$.

Example 18. Let $(S, +, \cdot) = (\mathbb{Z}_+, \oplus, \odot)$ be a commutative semiring, where $a \oplus b := gcd\{a, b\}$ and $a \odot b := lcm\{a, b\}$, for a and b in $\mathbb{Z}_+$. Let

$$A = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{pmatrix} \in M_3(S)$$

We find out that all the ranks of A, $(t(A), z(A), rank(A), mc(A), mr(A)$) are as in the example $(\mathbb{Z}_+, +, \cdot)$, but the difference is in the column rank of A. To show that in the following: The column rank ($c(A)$): It is easy to see that the linear span of the columns of A is

$$\mathscr{L}\left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\} = \mathscr{L}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\}.$$

Namely, $3 \oplus 4 = 1$. So the column rank of A is equal to two, i.e., $c(A) = 2$.

The spanning column rank of A ($sc(A)$). Note that

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \notin \mathscr{L}\left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\}$$

Namely, if

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = m \odot \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \oplus n \odot \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} m \odot 4 \oplus n \odot 0 \\ m \odot 0 \oplus n \odot 3 \\ m \odot 0 \oplus n \odot 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} lcm\{m, 4\} \\ lcm\{3, n\} \\ lcm\{3, n\} \end{pmatrix}$$

But, $3 \neq lcm\{m, 4\}$ because $4 \nmid 3$. Similarly,

$$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \notin \mathscr{L} \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \notin \mathscr{L} \left\{ \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\} = \mathscr{L} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\},$$

hence $sc(A) = 3$.

5.2 Permanent rank

In this section we introduce a new rank for matrices over semirings, based on permanents. Recall that for $A \in M_n(S)$, the permanent of A, denoted by $\mathrm{per}(A)$ is defined by

$$\mathrm{per}(A) := \sum_{\sigma \in \mathcal{S}_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}.$$

Definition 28. *Let $A \in M_n(S)$. For each $k = 1, \ldots, n$, $I_k(A)$ will denote the ideal in S generated by all permanents of $k \times k$ submatrices of A.*

Thus, to compute $I_k(A)$, calculate the permanents of all $k \times k$ submatrices of A and then find the ideal of S these permanents generate. Laplace expansion, which also holds for semirings, implies that permanents of $(k + 1) \times (k + 1)$ submatrices of A lie in $I_k(A)$. Thus, we have the following chain of ideals in S:

$$I_n(A) \subseteq I_{n-1}(A) \subseteq \cdots \subseteq I_2(A) \subseteq I_1(A) \subseteq I_0(A) = S.$$

It will be notationally convenient to extend the definition of $I_k(A)$ to all values of $k \in \mathbb{Z}$ as follows:

$$I_k(A) = \begin{cases} (0), & \text{if } k > n \\ S, & \text{if } k \leq 0. \end{cases}$$

Then we have

$$(0) = I_{n+1}(A) \subseteq I_n(A) \subseteq \cdots \subseteq I_1(A) \subseteq I_0(A) = S.$$

We can now consider this sequence of ideals. Computing the annihilator of each ideal in this sequence, we get the following chain of ideals.

$$(0) = \mathrm{Ann}_S(S) \subseteq \mathrm{Ann}_S(I_1(A)) \subseteq \mathrm{Ann}_S(I_2(A)) \subseteq \cdots \subseteq \mathrm{Ann}_S(I_n(A)) \subseteq \mathrm{Ann}_S(I_{n+1}(A)) = S.$$

Notice that if $\mathrm{Ann}_S(I_k(A)) \neq (0)$, then $\mathrm{Ann}_S(I_r(A)) \neq (0)$ for all $r \geq k$. Thus, the following definition makes perfectly good sense.

Definition 29. *The permanent rank of a matrix $A \in M_n(S)$, denoted by* $\mathrm{permrank}(A)$*, is the maximum integer k such that the annihilator of the ideal generated by all permanents of $k \times k$ submatrices of A is zero, i.e.,*

$$\mathrm{permrank}(A) = \max\{k : \mathrm{Ann}_S(I_k(A)) = (0)\}$$

Basic properties of this rank function are determined in the following theorem.

Theorem 7. *Let $A \in M_n(S)$.*

(a) $0 \leq \text{permrank}(A) \leq n$.

(b) $\text{permrank}(A) = \text{permrank}(A^t)$.

(c) $\text{permrank}(A) = 0$ *if and only if* $\text{Ann}_S(I_1(A)) \neq (0)$.

(d) $\text{permrank}(A) < n$ *if and only if per*$(A) \in Z(S)$*, the set of all zero divisors in S*

Proof. (a) $I_0(A) = S$, and $\text{Ann}_S(S) = (0)$. Thus, $\text{permrank}(A) \geq 0$. On the other hand, if $k > n$, then $I_k(A) = (0)$ and $\text{Ann}_S((0)) = S$. Therefore, $\text{permrank}(A) \leq n$.

(b) Since we have $I_\alpha(A) = I_\alpha(A^t)$ for all $\alpha \in \mathbb{Z}$, then $\text{Ann}_S(I_\alpha(A)) = \text{Ann}_S(I_\alpha(A^t))$, hence $\text{permrank}(A) = \text{permrank}(A^t)$.

(c) Suppose $\text{permrank}(A) = 0$. That means $\max\{k : \text{Ann}_S(I_k(A)) = (0)\} = 0$. So $k = 0$ is maximum of all k such that $\text{Ann}_S(I_k(A)) = (0)$.That means for all $k > 0$ we have $\text{Ann}_S(I_k(A)) \neq (0)$. Hence $\text{Ann}_S(I_1(A)) \neq (0)$. Conversely, assume that $\text{Ann}_S(I_1(A)) \neq (0)$. Since $\text{Ann}_S(I_1(A)) \subseteq \text{Ann}_S(I_2(A)) \subseteq \cdots \subseteq \text{Ann}_S(I_n(A)) \subseteq \text{Ann}_S(I_{n+1}(A)) = \text{Ann}_S((0)) = S$, then for all $k > 1$, $\text{Ann}_S(I_k(A)) \neq 0$, but $\text{Ann}_S(I_0(A)) = (0)$. Hence $\text{permrank}(A) = 0$.

(d) Suppose $\text{per}(A) \in Z(S)$. So, there exists $s \in S$ such that, $\text{per}(A) \cdot s = 0$. Therefore, $s \in \text{Ann}_S(I_n(A))$ and $\text{Ann}_S(I_n(A)) \neq 0$. It follows that $\text{permrank}(A) < n$. Conversely, assume that $\text{permrank}(A) < n$. That means $\max\{k : \text{Ann}_S(I_k(A)) = (0)\} < n$, i.e., $\text{Ann}_S(I_n(A)) \neq 0$. It follows that there exists $s \in \text{Ann}_S(I_n(A)) \backslash \{0\}$. So, $s \cdot \text{per}(A) = 0$ and $\text{per}(A) \in Z(S)$. □

Corollary 2. *Let $A \in M_n(S)$. Then if per$(A) \in U(S)$ then* $\text{permrank}(A) = n$.

Proof. Suppose $per(A) \in U(S)$, so $per(A) \notin Z(S)$. From (e) in last theorem this implies that $\text{permrank}(A) = n$. □

We will discuss examples of the permrank for matrices over commutative semirings with lots of zero divisors. Let X be any nonempty set. We use the commutative semiring $(S, +, \cdot) = (P(X), \cup, \cap)$. So, in the following, a, b, c, d are subsets of X. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(S),$$

such that $a, b, c, d \subseteq X$. We will compute $I_k(A), k = 0, 1, 2$. $I_0(A) = S, I_1(A) = \langle a, b, c, d\rangle$, $I_2(A) = \langle ad + bc\rangle$ (we denote by $\langle a_1, \ldots, a_n\rangle$ the ideal in S generated by the elements $a_1, \ldots, a_n$.

We have $\langle ad+bc\rangle \subseteq \langle a, b, c, d\rangle$, i.e., $I_2(A) \subseteq I_1(A)$, and $\mathrm{Ann}_S(\langle a, b, c, d\rangle) \subseteq \mathrm{Ann}_S(\langle ad+bc\rangle)$. Note that, $z \in \mathrm{Ann}_S(\langle a, b, c, d\rangle)$ if and only if

$$
\begin{aligned}
z \cdot a &= 0 \Leftrightarrow z \cap a = \emptyset \\
z \cdot b &= 0 \Leftrightarrow z \cap b = \emptyset \\
z \cdot c &= 0 \Leftrightarrow z \cap c = \emptyset \\
z \cdot d &= 0 \Leftrightarrow z \cap d = \emptyset
\end{aligned}
$$

If there is $z \neq \emptyset$ satisfying these conditions, then $a \cup b \cup c \cup d \neq X$. Namely, If $a \cup b \cup c \cup d = X$, then, since $z \subseteq X$,

$$
\begin{aligned}
z &= z \cap X \\
&= z \cap (a \cup b \cup c \cup d) \\
&= (z \cap a) \cup (z \cap b) \cup (z \cap c)\,(z \cap d) \\
&= \emptyset \cup \emptyset \cup \emptyset \cup \emptyset \\
&= \emptyset.
\end{aligned}
$$

So, $\mathrm{Ann}_S(\langle a, b, c, d\rangle) \neq \{0\} \Leftrightarrow a \cup b \cup c \cup d \neq X$. Also, $\mathrm{Ann}_S(\langle ad+bc\rangle) \neq \{0\} \Leftrightarrow ad+bc \neq X$. So if $a \cup b \cup c \cup d \neq X$, $\mathrm{permrank}(A) = 0$. If $a \cup b \cup c \cup d = X$, and $ad + bc \neq X$, then $\mathrm{permrank}(A)) = 1$. Finally if $ad + bc = X$, $\mathrm{permrank}(A) = 2$.

We discussed the permrank for matrices in $M_2(S)$, and we will discuss the permrank for some matrices in $M_3(S)$. Let

$$
A = \begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix} \in M_3(S),
$$

where $(S,+,\cdot) = (P(X),\cup,\cap)$, X is any set, and $a,b,c \subseteq X$. We will compute $I_k(A), k = 0,1,2,3$. $I_0(A) = S$, $I_1(A) = \langle a,b,c\rangle$,

$I_2(A) = \langle a{+}b, ab{+}bc, b{+}ac, ab{+}bc, a{+}c, ab{+}bc, b{+}ac, ba{+}cb, a{+}b\rangle = \langle a{+}b, a{+}c, b{+}ac\rangle$,

$$I_3(A) = \langle a+bc+cb+ac+ba+ab\rangle = \langle a+bc+ac+ab\rangle = \langle a+bc\rangle.$$

We explain how to compute $I_3(A)$ in the following:

$$\begin{aligned}
\operatorname{per}\begin{pmatrix} a & b & c \\ b & a & b \\ c & b & a \end{pmatrix} &= a\cdot\operatorname{per}\begin{pmatrix} a & b \\ b & a \end{pmatrix} + b\cdot\operatorname{per}\begin{pmatrix} b & b \\ c & a \end{pmatrix} + c\cdot\operatorname{per}\begin{pmatrix} b & a \\ c & b \end{pmatrix} \\
&= a\cdot\left(a^2+b^2\right) + b\cdot(ba+bc) + c\cdot\left(b^2+ac\right) \\
&= a+ab+ab+bc+bc+ac,
\end{aligned}$$

since $ac = a\cap c \subseteq a$, and $a+ac = a\cup(a\cap c) = a$ then $I_3(A) = (a+bc)$. We have

$$(a+bc) \subseteq (a+b, a+c, b+ac) \subseteq (a,b,c),$$

i.e., $I_3(A) \subseteq I_2(A) \subseteq I_1(A)$, and

$$\mathrm{Ann}_S(\langle a,b,c\rangle) \subseteq \mathrm{Ann}_S(\langle a+b, a+c, b+ac\rangle) \subseteq \mathrm{Ann}_S(\langle a+bc\rangle).$$

We have

$$\mathrm{Ann}_S(\langle a,b,c\rangle) \neq \{0\} \Leftrightarrow a\cup b\cup c \neq X$$

Note,

$$\begin{aligned}
z \in \mathrm{Ann}_S(\langle a,b,c\rangle) \Leftrightarrow z\cdot a &= 0 \\
z\cdot b &= 0 \\
z\cdot c &= 0.
\end{aligned}$$

If $a\cup b\cup c \neq X$, then there exists $x \in X\setminus(a\cup b\cup c)$. We take $z = \{x\}$, then $z\cap a = \emptyset, z\cap b = \emptyset, z\cap c = \emptyset$. If $a\cup b\cup c = X$, then $\mathrm{Ann}_S(\langle a,b,c\rangle) = \{0\}$. Also,

$$\begin{aligned}
z \in \mathrm{Ann}_S(\langle a+b, a+c, b+ac\rangle) \quad \Leftrightarrow \quad & z\cdot(a+b) = 0 \\
& z\cdot(a+c) = 0 \\
& z\cdot(b+ac) = 0.
\end{aligned}$$

$$\begin{aligned}\mathrm{Ann}_S(\langle a+b, a+c, b+ac\rangle) \neq \{0\} \Leftrightarrow (a+b)+(a+c)+(b+ac) &\neq X\\ a+b+c+ac &\neq X\\ a+b+c &\neq X\end{aligned}$$

$$\mathrm{Ann}_S(\langle a+bc\rangle) \neq \{0\} \Leftrightarrow a+bc \neq X.$$

Now, we apply that on a numerical example.

Example 19. Let $X = \{1, 2, 3, 4, 5\}$, $a = \{1, 2, 3\}$, $b = \{3, 4\}$, $c = \{4, 5\}$. We see that $a \cup b \cup c = X \Rightarrow \mathrm{Ann}_S(\langle a, b, c\rangle) = \{0\}$, $\mathrm{Ann}_S(\langle a+b, a+c, b+ac\rangle) = \{0\}$, but $\mathrm{Ann}_S(\langle a+bc\rangle) \neq \{0\}$, because $a \cup (b \cap c) = \{1, 2, 3\} \cup \{4\} \neq X$, so there is $z \in \mathrm{Ann}_S(\langle a+bc\rangle)$ such that $z \in X \setminus (a \cup b \cup c)$, i.e., $z = \{5\}$ so $\mathrm{Ann}_S(\langle a+bc\rangle) \neq \{0\}$, i.e., $I_3(A) \neq \{0\}$. Hence, $\mathrm{permrank}(A) = 2$.

Example 20. Let $S = \mathbb{Z}/6\mathbb{Z} = \{0, 1, 2, 3, 4, 5\}$.

(a) Suppose

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} \in M_n(S)$$

Clearly A is a nonzero matrix, and every entry in A is a zero divisor in S. $I_2(A) = \langle 4\rangle = 4S$, $I_1(A) = \langle 0, 2\rangle = 2S$, and $\mathrm{Ann}_S(I_2(A)) = \mathrm{Ann}_S(4S) = 3S \neq (0)$, $\mathrm{Ann}_S(I_1(A)) = \mathrm{Ann}_S(2S) = 3S \neq (0)$. Thus, $\mathrm{permrank}(A) = 0$ (by using (c) of the theorem).

(b) Let

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \in M_n(S)$$

Every entry in B is a zero divisor in S. Since $\mathrm{per}(B) = (0)$, (d) implies $\mathrm{permrank}(B) < 2$. Since $I_1(B) = \langle 0, 2, 3\rangle = 2S + 3S = S$, $\mathrm{Ann}_S(I_1(B)) = \mathrm{Ann}_S(S) = (0)$, so from (d) $\mathrm{permrank}(B) \neq 0$. Therefore $\mathrm{permrank}(B) = 1$.

(c) Suppose

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \in M_n(S)$$

We have $\mathrm{per}(C) = 5 \in U(S)$. Then $\mathrm{permrank}(C) = 2$ by Corollary 2.

Bibliography

[1] Javed Ahsan, John N. Mordeson, Muhammad Shabir, Fuzzy Semirings with Applications to Automata Theory, Studies in Fuzziness and Soft Computing: vol. 278, Springer, Berlin, 2012.

[2] W. C. Brown, Matrices over commutative rings, Marcel Dekker, Inc. (1993)

[3] P. Butkovič, Max-algebra: the linear algebra of combinatorics? Linear Algebra Appl. 367 (2003) 313–335.

[4] R. A. Cuninghame-Green, Minimax Algebra, Lecture notes in Economics and Mathematical Systems, vol. 166, Springer, Berlin, 1979.

[5] R. A. Cunninghame-Green, P. Butkovič, Bases in max-algebra, Linear Algebra Appl. 389 (2004) 107–120.

[6] S. Ghosh (1996) Matrices over Semirings, *Information Sciences* **90**, 221–230.

[7] J. S. Golan, Semirings and Their Applications, Kluwer Academic Publishers, Dordrecht/Boston/London, 1999.

[8] Alexander Guterman, Rank and determinant functions for matrices over semirings, London Mathematical Society Lecture Notes 347 (2007), 1–33.

[9] Asmaa M. Kanan, Zoran Z. Petrović, Note on cardinality of bases in semilinear spaces over zerosumfree semirings, Linear Algebra Appl. 439(2013), no. 10, 2795–2799.

[10] Asmaa M. Kanan, Zero divisors for matrices over commutative semirings, ScienceAsia, to appear.

[11] P. Moller, Theorie algèbrique des systèmes a évènements discrets, Thèse de doctorat, Paris, 1988.

[12] A. Di Nola, A. Lettieri, I. Perfilieva, V. Novák, Algebraic analysis of fuzzy systems, Fuzzy Sets and Systems 158 (2007) 1–22.

[13] C. Reutenauer, H. Straubing (1984) Inversion of Matrices over a Commutative Semiring, *Journal of Algebra* **88**, 350–360.

[14] Qian-yu Shu, Xue-ping Wang, Bases in semilinear spaces over zerosumfree semirings, Linear Algebra Appl. 435 (2011) 2681–2692.

[15] S. Sombatboriboon, W. Mora, Y. Kemprasit (2011) Some results concerning invertible matrices over semirings, *ScienceAsia* **37**, 130–135.

[16] Y. J. Tan, On invertible matrices over antirings, Linear Algebra Appl. 423 (2007) 428–444.

[17] E. Wagneur, Moduloïds and pseudomodules. 1. Dimension theory, Discrete Math. 98 (1991) 57–73.

[18] Shan Zhao, Xue-ping Wang, Invertible matrices and semilinear spaces over commutative semirings, Inform. Sci. 180 (2010) 5115–5124.

[19] Shan Zhao, Xue-ping Wang, Bases in semilinear spaces over join-semirings, Fuzzy Sets and Systems 182 (2011) 93–100. ‘

Biografija autora

Asma M. Kanan je rodjena 12. 06. 1979. godine u Sabrati u Libiji. Godine 2001. završila je osnovne studije na Departmanu za matematiku u okviru Fakulteta za nauke Univerziteta 7. april u Zaviji u Libiji. 2005. godine je stekla zvanje mastera na istom departmanu. Od 2001. do 2005. godine je radila kao asistent na Univerzitetu 7. april u Zaviji, a od 2005. do 2008. kao predavač na istom univerzitetu. Držala je nastavu iz sledećih oblasti: Linearna algebra, Analitička geometrija, Prostorna geometrija, Obične diferencijalne jednačine, Kompleksna analiza, Diferencijalna geometrija i Linearno programiranje. Doktorske studije na Matematičkom fakultetu u Beogradu je započela u školskoj 2009/10 godini.

Прилог 1.

Изјава о ауторству

Потписани Asmaa Kanan

број индекса 2056/2009

Изјављујем

да је докторска дисертација под насловом

O deliteljima nule, invertibilnosti i rangu matrica nad komutativnim poluprstenima

- резултат сопственог истраживачког рада,
- да предложена дисертација у целини ни у деловима није била предложена за добијање било које дипломе према студијским програмима других високошколских установа,
- да су резултати коректно наведени и
- да нисам кршио/ла ауторска права и користио интелектуалну својину других лица.

Потпис докторанда

У Београду, 14.11.2013

Прилог 2.

Изјава о истоветности штампане и електронске верзије докторског рада

Име и презиме аутора Asmaa Kanan

Број индекса 2056/2009

Студијски програм Matematika

Наслов рада O deliteljima nule, invertibilnosti i rangu matrica nad komutativnim poluprstenima

Ментор Zoran Petrović

Потписани/а Asmaa Kanan

Изјављујем да је штампана верзија мог докторског рада истоветна електронској верзији коју сам предао/ла за објављивање на порталу **Дигиталног репозиторијума Универзитета у Београду.**

Дозвољавам да се објаве моји лични подаци везани за добијање академског звања доктора наука, као што су име и презиме, година и место рођења и датум одбране рада.

Ови лични подаци могу се објавити на мрежним страницама дигиталне библиотеке, у електронском каталогу и у публикацијама Универзитета у Београду.

Потпис докторанда

У Београду, 14.11.2013.

Прилог 3.

Изјава о коришћењу

Овлашћујем Универзитетску библиотеку „Светозар Марковић“ да у Дигитални репозиторијум Универзитета у Београду унесе моју докторску дисертацију под насловом:

O deliteljima nule, invertibilnosti i rangu matrica nad komutativnim poluprstenima

која је моје ауторско дело.

Дисертацију са свим прилозима предао/ла сам у електронском формату погодном за трајно архивирање.

Моју докторску дисертацију похрањену у Дигитални репозиторијум Универзитета у Београду могу да користе сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons) за коју сам се одлучио/ла.

1. Ауторство
2. Ауторство - некомерцијално
3. Ауторство – некомерцијално – без прераде
4. Ауторство – некомерцијално – делити под истим условима
5. Ауторство – без прераде
6. Ауторство – делити под истим условима

(Молимо да заокружите само једну од шест понуђених лиценци, кратак опис лиценци дат је на полеђини листа).

Потпис докторанда

У Београду, 14.11.2013.

1. Ауторство - Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, и прераде, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце, чак и у комерцијалне сврхе. Ово је најслободнија од свих лиценци.

2. Ауторство – некомерцијално. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, и прераде, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце. Ова лиценца не дозвољава комерцијалну употребу дела.

3. Ауторство - некомерцијално – без прераде. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, без промена, преобликовања или употребе дела у свом делу, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце. Ова лиценца не дозвољава комерцијалну употребу дела. У односу на све остале лиценце, овом лиценцом се ограничава највећи обим права коришћења дела.

4. Ауторство - некомерцијално – делити под истим условима. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, и прераде, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце и ако се прерада дистрибуира под истом или сличном лиценцом. Ова лиценца не дозвољава комерцијалну употребу дела и прерада.

5. Ауторство – без прераде. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, без промена, преобликовања или употребе дела у свом делу, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце. Ова лиценца дозвољава комерцијалну употребу дела.

6. Ауторство - делити под истим условима. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, и прераде, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце и ако се прерада дистрибуира под истом или сличном лиценцом. Ова лиценца дозвољава комерцијалну употребу дела и прерада. Слична је софтверским лиценцама, односно лиценцама отвореног кода.