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COURANT ALGEBROIDS IN BOSONIC STRING THEORY

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KURANTOVI ALGEBROIDI U BOZONSKOJ TEORIJI STRUNA

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Abstract

Generalized geometry is a new mathematical paradigm in which vectors and 1-forms are united and investigated as single objects - generalized vectors. In this dissertation, we explore symmetries of bosonic string theory and their relations with T-duality in the formalism of generalized geometry. The generator of both diffeomorphisms and local gauge transformations is constructed and expressed as an $O(D, D)$ invariant inner product of two generalized vectors. In the same way that the Poisson bracket algebra of generators of diffeomorphism gives rise to the Lie bracket, the algebra of the extended generators gives rise to the Courant bracket. Taking into account the T-duality relation between two string symmetries, we interpret the Courant bracket as the T-dual extension of the Lie bracket [1].

We then develop a simple procedure for twisting the Courant bracket with any $O(D, D)$ transformation, allowing us to obtain Courant brackets deformed with different fluxes. The crux of this method consists of expressing the generator in the basis of non-canonical currents, which are connected with canonical variables via the $O(D, D)$ transformation. We show that the Poisson bracket algebra of generators in the basis of currents closes on the appropriate twisted Courant bracket. We prove that there is a natural way to define a Courant algebroid using these twisted Courant brackets. We provide many examples of $O(D, D)$ transformations and their corresponding twisted Courant brackets, including the B-twisted Courant bracket and the θ-twisted Courant bracket. The B-twisted Courant bracket is characterized by H flux appearing in the algebra of currents, while the θ-twisted Courant bracket is characterized by the so-called non-geometric Q and R fluxes. It has been shown that these brackets are mutually T-dual [2].

In addition, we construct the generator that produces the Courant bracket twisted simultaneously by B and θ in its Poisson bracket algebra. This generator is expressed in terms of currents that contain all string fluxes in their Poisson bracket relations. Moreover, we show that the Courant bracket twisted simultaneously by B and θ is invariant under the T-duality [3]. We also demonstrate that all fluxes can exist on the Dirac structures associated with the Courant algebroid for this bracket, without any restrictions imposed on fluxes.

In the end, results are generalized to a double theory, in which variables depend on both initial and T-dual coordinates. The algebra of generators that include both initial and T-dual diffeomorphisms closes on the double field extension of the Courant bracket called C-bracket. Following the same procedure as in the single theory, we obtained the B-twisted and θ-twisted C-brackets [4]. We demonstrate that by projecting the twisted C-brackets to the initial and T-dual phase spaces, the mutually T-dual twisted Courant brackets are obtained.

Key words: Bosonic string, T-duality, Generalized geometry
Scientific field: Physics
Research area: String theory

Sažetak

Generalisana geometrija podrazumeva novu matematičku paradigmu u kojoj se vektori i 1-forme objedinjuju i razmatraju kao jedinstveni objekti - generalisani vektori. U ovoj disertaciji istražujemo simetrije bozonske teorije struna i njihove veze sa T-dualnošću korišćenjem formalizma generalisane geometrije. Konstruisan je jedinstven generator difeomorfizama i lokalnih gradijentnih transformacija i predstavljen kao $O(D, D)$ invarijantan skalarni proizvod između dva generalisana vektora. Na isti način kao što u algebri Poasonovih zagrada generatora difeomorfizama nastaje Lijeva zagradi, algebra proširenog generatora daje Kurantovu zagradu. Uzimajući u obzir T-dualne veze između ove dve simetrije, Kurantova zagrada je interpretirana kao ekstenzija Lijeve zagrade invarijantna na T-dualnost [1].

Zatim razvijamo jednostavnu proceduru za pronalaženje Kurantovih zagrada zavrnutih proizvoljnim $O(D, D)$ transformacijama, što nam omogućava da dobijemo Kurantove zagrade deformisane različitim fluksevima. Osnova metode je predstavljanje generatora u bazisu nekanonskih struja, koje su $O(D, D)$ transformacijom povezane sa kanonskim promenljivama. Pokazano je da se algebra Poasonovih zagrada između generatora izraženih preko struja zatvara na odgovarajućoj zavrnutoj Kurantovoj zagradi. Dokazano je i da takva zavrnuta Kurantova zagrada definiše na prirodan način Kurantov algebroid. Dali smo više primera $O(D, D)$ transformacija i odredili njima odgovarajuće zavrnute Kurantove zagrade, uključujući i B-zavrnutu i θ-zavrnutu Kurantovu zagradu. Kurantovu zagradu zavrnutu poljem B karakteriše pojavljivanje H fluksa u algebri struja, dok Kurantovu zagradu zavrnutu poljem θ karakteriše pojavljivanje takozvanih negeometrijskih Q i R flukseva. Pokazano je da su ove dve zagrade međusobno T-dualne [2].

Dodatno, konstruisan je i generator koji daje Kurantovu zagradu istovremeno zavrnutu poljima B i θ. Ovaj generator izražen je preko pomoćnih struja u čijim algebarskim relacijama izraženim preko Poasonovih zagrada se dobijaju svi fluksevi teorije struna. Dodatno, pokazali smo da je na ovakav način zavrnuta Kurantova zagrada i invarijantna na T-dualnost [3]. Takođe smo pokazali da svi fluksevi mogu postojati na Dirakovim strukturama Kurantovog algebroida definisanog ovom zagradom, bez ikakvih ograničenja na tim fluksevima.

Na kraju, uopštili smo rezultate na duplu teoriju, u kojoj sve promenljive zavise i od početnih i od T-dualnih koordinata. Algebra generatora koji obuhvata difeomorfizme i T-dualne difeomorfizme zatvara se na C-zagradi, što je generalizacija Kurantove zagrade na dupli fazni prostor. Koristeći se istom procedurom kao i u nedupliranoj teoriji, dobili smo C-zagrade zavrnute poljima B i θ [4]. Projektujući ove zagrade na međusobno T-dualne fazne prostore, dobili smo međusobno T-dualne zavrnute Kurantove zagrade.

Ključne reči: Bozonska struna, T-dualnost, Generalisana geometrija
Naučna oblast: Fizika
Uža naučna oblast: Teorija struna

Contents

Part I

String theory essentials

Chapter 1

Introduction

Relation between physics and mathematics is a long and close one. Since Newton's publication of Philosophiæ Naturalis Principia Mathematica, the accepted paradigm was that laws of physics should have a precise mathematical formulation. Newton's three laws of motion formulated in Principia require only a simple calculus. Physical properties of the system, such as momentum, energy or angular momentum, are modeled by smooth functions. It was later demonstrated that the introduction of Lagrangian in configuration space is suitable for deriving equations of motions from the principle of minimal action. Equivalently, everything could be derived from the Hamiltonian in the phase space, obtained as a Legendre transformation of the Lagrangian. The need for more complex mathematical formalism grew with the expansion of knowledge of physical phenomena. In the formulation of quantum mechanics, physical variables are expressed as operators on the Hilbert space, and their values are the eigenvalues of these operators. For the formulation of general relativity, physicists incorporated Riemannian geometry.

Classical mechanics is able to predict the motion of many different objects, from billiard balls to rocket ships. Given the initial conditions, i.e. initial coordinates and velocities, the motion of any object is fully determined by the action of its forces. However, when it comes to the description of objects on very short or very large scales, classical mechanics is unable to make any meaningful predictions. It is important to point out that these extremes in scales brought fundamental changes in our understanding of space-time and challenged intuitive but wrong premises about the world around us.

Quantum theory is fundamentally predicated upon different premises than classical mechanics. For instance, Heisenberg's uncertainty principle disproves the assumption that coordinate and velocity can be simultaneously well-defined and expressed by single values. Waves and particles are not seen as fundamentally different phenomena, but each particle has a wave property. The energy of light is not continuous but is divided into a finite number of chunks of energy. These ideas were necessary to describe real physics phenomena, like the emission of electrons when metal is hit by light. What followed was the development of the quantum field theory, where particles arise as excitations of the fields, and their interactions are described by the coupling of their fields. This is all well combined into the Standard Model, which adequately describes electromagnetic, weak, and strong nuclear interaction, as was confirmed with a myriad of experiments on different energy scales with very high precision [5]. Though impressive, the Standard Model has a major limitation: it does not include gravity.

Gravity is best described by Einstein's general theory of relativity. It is formulated in a curved background, and gravity arises as a consequence of the space-time geometry. Many experiments confirmed Einstein's theory [6], including the light bending and time dilatation near a massive body, as well as recent observations of gravitational waves by LIGO [7, 8]. Gravity is a weak force, 25 orders of magnitude weaker than weak nuclear interactions, and even more times weaker than strong nuclear and electromagnet interactions. As such, on smaller scales, its effects are usually negligible. However, on large scales, when we are considering massive objects such as galaxies and black holes, it is the dominant force, due to its long range, and in these situations, the general theory of relativity applies. However, some situations do not quite fit in either of the descriptions. On very large energies, in the interior of black holes, or in the moments right after the Big Bang, we do not have a proper argument to rule out either relativistic or quantum effects. Therefore, it is believed that quantum mechanics and general relativity are effective theories of a more general quantum theory of gravity.

There is almost a universal belief that the quantum theory of gravity exists. The electromagnetic and weak interactions are unified at larger energies, and by symmetry breaking mechanism they separate. So the argument goes that on even higher energies all interactions should be unified. Moreover, the quantum field theory is formulated in a fixed background, while the nature of general relativity is that space-time itself is dynamic, so a more universal theory has to exist.

However, the formulation of this unified theory turned out to be challenging. If one proceeds with the quantization of gravity in the framework of perturbative quantum field theory, the theory is not renormalizable by the standard renormalization techniques. Moreover, the quantum effects of gravity are expected to appear at energies of $10^{19} GeV$, which are inaccessible with current accelerators, making experimental testing of quantum gravity very difficult. It can be argued that understanding the quantum theory of gravity will bring a better understanding of space-time and the fundamental properties of the world around us, in the same way as quantum field theory and general relativity did. As it was historically true, we can expect that description of the theory of quantum gravity will also require new areas of mathematics.

1.1 String theory

String theory [9, 10, 11] is a theoretical framework that postulates that the basic elements of Nature are one-dimensional strings, rather than point particles. Unlike Standard Model, which requires nineteen parameters to be obtained experimentally and put into theory by hand, the string theory requires only one dimensionless parameter, which is the fundamental string length. The quantized string has a discrete spectrum of vibrating modes, with different modes appearing as different particles at distances larger than the fundamental string length. The string propagating in space-time spans the two-dimensional world-sheet parametrized with a time-like parameter τ and space-like parameter σ, analogous to a relativistic point particle moving along the world-line. The world-sheet can either be a torus, in which case we have a closed string, or a strip, in which case we have an open string. In the former case, the periodic boundary conditions are imposed on the string target space. In the latter case, we can impose either Neumann conditions, where end-points of an open string are fixed in space, or Dirichlet boundary conditions, where they are on a dynamical object, called D-brane. While it might appear as having odd assumptions, there are good reasons that make string theory the candidate for the universal description of all interactions.

First of all, one of the string vibration modes produces a massless spin 2 particle, which has never been observed. Though initially, this led to the abandonment of the theory, in 1974 it has been shown that such a particle obeys the Ward identities and can be interpreted as graviton [12]. Formulation of the quantum theory of gravity is arguably the biggest challenge in contemporary physics, and the fact that string theory predicts a consistent quantum theory of gravity makes it appealing for research. Moreover, for one-loop corrections, the gravity emerging from string theory is renormalizable [13]. In particle physics, Feynman diagrams are webs of world lines that juxtapose in points, which are sources of singularities. In string theory, Feynman diagrams are two-dimensional surfaces, that intersect on smooth areas, and as such there are no local singularities. The divergences in string gravity cancel each other out. The fundamental string length provides a natural ultraviolet cut-off for graviton scattering amplitudes.

The first string theory that was developed is the bosonic string theory. Besides graviton, bosonic string theory also includes gauge bosons. It is an incomplete theory since it does not include fermions. Moreover, it predicts the existence of tachyons, particles of negative energy. These issues are resolved with the introduction of supersymmetry, leading to the development of superstring theory. Given that all particles appear in the spectrum of supersymmetric strings, the superstring theory is a promising candidate for a unified theory of all fundamental interactions. There are five different superstring theories that are anomaly free. These are type I, IIA, IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$.

As was the case with previous breakthroughs in theoretical physics, string theory is also enriched with more complicated mathematical apparatus. The world-sheet poses conformal symmetry. The conformal invariance on the quantum level imposes the number of dimensions of space-time. In bosonic string theory, the critical number of dimensions of space-time is twenty-six, while for the superstring theories, it is ten. From the optimistic perspective, one can claim that this is an advantage of string theory because the number of dimensions is predicted by the theory. From the pessimistic perspective, we do not observe ten dimensions and this is a challenge that the theory has to resolve.

Interestingly enough, the idea that all interactions can be unified in space-time with extra dimensions is older than the string theory itself. Einstein's general relativity and Maxwell equations were derived from gravity action in five dimensions in the works of Kaluza and Klein [14, 15]. They proposed that an additional dimension is compactified on a circle. This motivated the string theorists to consider the approach where supplementary dimensions are compactified.

1.2 T-duality

The propagation of closed strings in space-time with one dimension being compactified to a circle led to the discovery of T-duality, a striking feature unique to string theory. It was observed that two string theories, one with a dimension compactified on a circle with radius R and the other one with a dimension compactified on a circle with radius proportional to $\frac{1}{R}$ have the same mass spectrum, and therefore are physically indistinguishable. The winding number, that is to say, the number of times a closed string winds around a compact dimension, in one theory represents the momentum number in its T-dual theory, and vice versa. This is a simple example of a more general string phenomenon, that two theories can be defined on backgrounds with different geometries, or even topologies, but still, predict the same physics.

Dualities are relations between different actions that lead to the same observable quantities that are an integral part of the string theory. For the closed string moving in the background characterized by constant fields, the procedure of obtaining T-dual theory was developed by Buscher [16]. The open string with Neumann boundary conditions is T-dual to open strings with Dirichlet boundary conditions. Moreover, the T-duality connects IIA and IIB superstring theories, and two types of heterotic string theories. Together with S-duality, which connects theories with a strong and weak coupling constant, T-duality connects different supersymmetric string theories with a single, eleven-dimensional M-theory. This was observed by Witten in 1995, marking the so-called second superstring revolution. Before that, it was unclear which superstring theory is to be preferred.

As backgrounds become more complicated, obtaining the T-dualization procedure becomes more challenging. After T-dualization one can obtain so-called non-geometric backgrounds, where background fields are non-local. Though many advances in understanding this intriguing feature, we still do not have the universal procedure of obtaining T-duality for string moving in the arbitrary background. Given its importance in relating different superstring theories, a better understanding of T-duality is one of the string theory priorities.

The T-duality gives rise to a plethora of geometries and topologies. Therefore, its universal formulation requires a general mathematical framework, which can include these different spaces. The promising candidate for such a framework is generalized geometry. It is the geometry of the generalized tangent bundle, which is just a direct sum of the tangent and cotangent bundle over a manifold. Vectors and 1-forms are combined into single objects, called generalized vectors. On the space of generalized vectors, there is a natural way to define both a symmetric and antisymmetric inner product. The former is invariant under the $O(D, D)$ transformations, which are transformations that govern T-duality.

Furthermore, the generalized tangent bundle is equipped with the Courant bracket. It can be understood as a generalization of the Lie bracket on the generalized tangent bundle. Unlike the Lie bracket, the Courant bracket does not satisfy the Leibniz rule and Jacobi identity, though there are sub-bundles on which it does satisfy both of them and can be seen as the bracket of Lie algebra. Initially, the Courant bracket was constructed as a double of Lie bialgebroid, which is just the ordered pair of two Lie algebroids on mutually dual vector bundles. Soon after its construction, it was observed that one of the Lie algebroids can be twisted by an exact 3-form, and a twisted Courant bracket was obtained. Subsequently, Roytenberg showed that both Lie algebroids can be twisted, and constructed what is known as the Roytenberg bracket. The additional terms that appeared due to twisting can be interpreted as string theory fluxes.

Fluxes in string theory appear among others in the context of background compactification [17, 18, 19], and generalized geometry [20, 21, 22]. The vacuum in the compactification background is degenerated. Different possible configurations are parametrized by moduli, that appear as massless scalar fields in the lower dimensional theory. These fields are problematic from a phenomenological point of view since they would carry long-range interactions that are unphysical. The problem can be resolved by introducing fluxes to the background [23, 24, 25]. The fluxes generate the potential that stabilizes the vacuum expectation value and gives mass to moduli.

1.3 Overview of the thesis

In this dissertation, we explore symmetries of bosonic string theory and their relations with T-duality in the formalism of generalized geometry. The generator of both diffeomorphisms and local gauge transformations is constructed and expressed as an $O(D, D)$ invariant inner product of two generalized vectors. In the same way that the Poisson bracket algebra of generators of diffeomorphism gives rise to the Lie bracket, the algebra of the extended generators gives rise to the Courant bracket. Taking into account the T-duality relation between two string symmetries, we interpret the Courant bracket as the T-dual extension of the Lie bracket [1].

We then develop a simple procedure for twisting the Courant bracket with any $O(D, D)$ transformation, allowing us to obtain Courant brackets deformed with different fluxes. The crux of this method consists of expressing the generator in the basis of non-canonical currents, which are connected with canonical variables via the $O(D, D)$ transformation. We show that the Poisson bracket algebra of generators in the basis of currents closes on the appropriate twisted Courant bracket. We prove that there is a natural way to define a Courant algebroid using these twisted Courant brackets. We provide many examples of $O(D, D)$ transformations and their corresponding twisted Courant brackets, including the B-twisted Courant bracket and the θ-twisted Courant bracket. The B-twisted Courant bracket is characterized by H flux appearing in the algebra of currents, while the θ-twisted Courant bracket is characterized by the so-called non-geometric Q and R fluxes. It has been shown that these brackets are mutually T-dual [2].

In addition, we construct the generator that produces the Courant bracket twisted simultaneously by B and θ in its Poisson bracket algebra. This generator is expressed in terms of currents that contain all string fluxes in their Poisson bracket relations. Moreover, we show that the Courant bracket twisted simultaneously by B and θ is invariant under the T-duality [3]. We also demonstrate that all fluxes can exist on the Dirac structures associated with the Courant algebroid for this bracket, without any restrictions imposed on fluxes.

In the end, results are generalized to a double theory, in which variables depend on both initial and T-dual coordinates. The algebra of generators that include both initial and T-dual diffeomorphisms closes on the double field extension of the Courant bracket called C-bracket. Following the same procedure as in the single theory, we obtained the B-twisted and θ-twisted C-brackets [4]. We demonstrate that by projecting the twisted C-brackets to the initial and T-dual phase spaces, the mutually T-dual twisted Courant brackets are obtained.

Chapter 2

Action for bosonic string

In this chapter, we will give a brief overview of the action for a relativistic particle, after which we will by analogy, construct the first action for a bosonic string. We will then provide a non-linear σ-model, that describes propagation of the closed bosonic string in coordinately dependent background fields.

2.1 Relativistic particle

Let us first consider a relativistic free particle, moving in a curved D-dimensional background. The movement of the particle sweeps the one-dimensional world-line parametrized with a time-like parameter τ. The action of a relativistic particle is proportional to the invariant length of its trajectory

$$S_0 = m \int \sqrt{G_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau \,, \tag{2.1}$$

where $G_{\mu\nu}$ is the metric of the background, x^μ represent the coordinates of the particle, and $\dot{x}^\mu$ their derivatives with respect to τ. The proportionality constant m is obtained from the dimensional analysis and represents the mass of the particle. Under the reparametrization of the world-line $\tau' = f(\tau)$ we have

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau'} \frac{\partial \tau'}{\partial \tau} = \dot{f} \frac{\partial x^\mu}{\partial \tau'} \,, \quad d\tau' = \dot{f} d\tau \,, \tag{2.2}$$

and therefore the action (2.1) remains invariant under the reparametrization of the world-line.

We are not quite satisfied with the action that features the square root, because it is impossible to quantize it by Fenyman's path integral formalism. Recall that in this formalism, the quantum mechanical propagator is obtained by integrating over different contributions of all paths in configuration space, with weights being expressed as $e^{-i\frac{S}{\hbar}}$. We can avoid integrating over square root by using another linear action that is equivalent to the action (2.1) on the classical level. With the introduction of an independent auxiliary field $e(\tau)$ on the world-line, we define the action

$$S = \frac{1}{2}\int d\tau\Big(\frac{1}{e}G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + m^2 e\Big)\,. \tag{2.3}$$

The variation of the action along e produces the equations of motions

$$e = \frac{1}{m}\sqrt{G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,, \tag{2.4}$$

on which the action (2.3) becomes (2.1). The variation of action along the coordinates x^μ provides the well-known equation of motion for a free relativistic particle along the geodesic, given by

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0\,, \tag{2.5}$$

where $\Gamma^\rho_{\mu\nu}$ are Christoffel symbols, given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}(G^{-1})^{\rho\sigma}\Big(\partial_\mu G_{\nu\sigma} + \partial_\nu G_{\mu\sigma} - \partial_\sigma G_{\mu\nu}\Big)\,. \tag{2.6}$$

In case of a relativistic particle moving in the electromagnetic field, one should add the interacting term to the action

$$S_{int} = \int d\tau q A_\mu \dot{x}^\mu\,, \tag{2.7}$$

where q is the electric charge of the particle, while A_μ is the vector potential. The action (2.7) is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu\lambda\,, \tag{2.8}$$

due to

$$\int d\tau q\partial_\mu\lambda\dot{x}^\mu = \int d\tau q\dot{\lambda} = 0\,. \tag{2.9}$$

2.2 Action for non-interacting string

Now consider a one-dimensional string and suppose we want to introduce its action by analogy with the relativistic particle. In the same way that the particle sweeps the world-line, a string sweeps the world-sheet. The Nambu-Goto action [26] that describes the string is proportional to the area of the worldsheet. It is given by

$$S_{NG} = \kappa\int_\Sigma d^2\xi\sqrt{-\det\Big(\partial_\alpha x^\mu\partial_\beta x^\nu G_{\mu\nu}\Big)}\,,\quad d^2\xi = d\sigma d\tau\,, \tag{2.10}$$

where $\kappa = \frac{1}{2\pi l_s^2}$, with l_s being the string length scale. It is the only parameter in string theory. The string is moving in a D-dimensional space-time characterized with a constant metric $G_{\mu\nu}$, where $\mu,\ \nu$ are the coordinates of the space-time $\mu,\ \nu \in 0,\ 1, ..., D-1$. The indices $\alpha,\ \beta = 0,\ 1$ are coordinates on the world-sheet Σ, parametrized with one time-like parameter $\xi^0 = \tau$ and one space-like parameter $\xi^1 = \sigma$. For closed strings, the topology of world-sheet is a torus $\mathbb{R} \times S^1$, where $-\infty \leq \tau \leq \infty$ and $0 \leq \sigma < 2\pi$.

Obviously, the same problem as with the action for relativistic particle (2.1) persists - the Nambu-Goto action cannot be quantized with the Feynman path integral procedure due to its nonlinearity. In an analogous way, it is possible to introduce the world-sheet metric $g_{\alpha\beta}$ and construct the Polyakov action

$$S_P = \frac{\kappa}{2} \int d^2\sigma \sqrt{-g}\, g^{\alpha\beta} G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{2.11}$$

where $g^{\alpha\beta}$ is the inverse of the world-sheet metric $g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$, and g is determinant of the world-sheet metric $g = \det g_{\alpha\beta}$. The Polyakov action is invariant under the global Poincaré transformations

$$x^{\mu\prime}(\xi) = \Lambda^\mu{}_\nu x^\nu(\xi) + a^\mu \,, \quad g'_{\alpha\beta} = g_{\alpha\beta} \,, \tag{2.12}$$

where a^μ are translation parameters, and $\Lambda^\mu{}_\nu$ are Lorentz transformations that satisfy $\Lambda^T \eta \Lambda = \eta$, for the Minkowski metric η. Additionally, the action is invariant under the reparametrization of the world-sheet

$$\xi^\alpha \to \xi^{\alpha\prime}(\xi) \,, \quad x^{\mu\prime}(\xi') = x^\mu(\xi) \,, \quad g'_{\alpha\beta}(\xi') = \frac{\partial \xi^\gamma}{\partial \xi'^\alpha} \frac{\partial \xi^\delta}{\partial \xi'^\beta} g_{\gamma\delta} \,, \tag{2.13}$$

as well as under the Weyl transformations

$$g'_{\alpha\beta}(\xi) = e^{\phi(\xi)} g_{\alpha\beta}(\xi) \,, \quad x^{\mu\prime}(\xi) = x^\mu(\xi) \,. \tag{2.14}$$

Reparametrization and Weyl transformations are local transformations and can be used to choose the gauge. The theory invariant under the Weyl transformations is said to be conformally invariant. The string theory is therefore the conformal field theory.

The variation of the Polyakov action (2.11) with respect to the worldsheet metric gives rise to the equations of motions, on which the Nambu-Goto action (2.10) is obtained. To demonstrate this, we will find useful the following relations

$$\delta g = -g g_{\alpha\beta} \delta g^{\alpha\beta} \,, \quad \delta\sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\alpha\beta} \delta g^{\alpha\beta} \,, \tag{2.15}$$

so that we can obtain the equations of motions by varying the Polyakov action with respect to the worldsheet metric $\frac{\delta S_P}{\delta g^{\alpha\beta}}$

$$\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu G_{\mu\nu} = 0 \,. \tag{2.16}$$

Its solution is in the form

$$g_{\alpha\beta} = \lambda\, \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}\,, \tag{2.17}$$

where λ is an arbitrary scalar, due to Weyl symmetry (2.14). Substituting (2.17) into the Polyakov action (2.11), one obtains the Nambu-Goto action (2.10). The second set of equations of motion is obtained from the action variation with respect to x^μ, in which case the wave equations are obtained

$$\partial^\alpha \partial_\alpha x^\mu = \ddot{x}^\mu - x''^\mu = 0\,. \tag{2.18}$$

If we introduce the light-cone coordinates by

$$\xi^\pm = \tau \pm \sigma\,, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)\,, \tag{2.19}$$

the equation of motion (2.18) can be rewritten as

$$\partial_+ \partial_- x^\mu = 0\,. \tag{2.20}$$

Now its general solution will have a decomposition to the left-movers x^μ_L and right-movers x^μ_R

$$x^\mu = x^\mu_L(\xi_+) + x^\mu_R(\xi_-)\,, \tag{2.21}$$

which can be expanded in modes by

$$\begin{aligned} x^\mu_L &= \frac{1}{2}x^\mu_0 + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + i\frac{l_s}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-2in(\tau+\sigma)} \\ x^\mu_R &= \frac{1}{2}x^\mu_0 + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + i\frac{l_s}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in(\tau-\sigma)}\,, \end{aligned} \tag{2.22}$$

where x^μ_0 is the center of mass position of a string, and p^μ is total string momentum. The exponential terms represent the string excitation modes, and α^μ_n and $\tilde{\alpha}^\mu_n$ its coefficients.

2.3 Bosonic string σ-model

The Polyakov action can be quantized, in which case oscillatory modes α^μ_n and $\tilde{\alpha}^\mu_n$ are promoted to operators that satisfy the harmonic oscillator algebra relations, so that they can be interpreted as creation and annihilation operators. The ground state is defined as the state annihilated by annihilation operators. Bosonic string theory also predicts the existence of negative norm states, which are removed in case of strings moving in 26-dimensional space-time. This is what we call a critical bosonic string theory [1]. In this case, the first excited state gives a set of $24^2 = 576$ of the states that correspond to the tensor products of two $SO(24)$ representations. It includes the symmetric traceless part, the antisymmetric part, and the trace.

The symmetric traceless part transforms as a massless particle of spin 2, and for that reason the background field corresponding to it is space-time metric $G_{\mu\nu}$. The antisymmetric part is represented by the Kalb-Ramond field $B_{\mu\nu}$, and the trace is represented with the scalar dilaton field Φ. The string σ-model is described by the following action:

$$S = \kappa \int_\Sigma d^2\xi \Big[\frac{1}{2}\sqrt{-g}g^{\alpha\beta}G_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu + \frac{\pi}{\kappa}\sqrt{-g}\Phi(x)R^{(2)}\Big] , \qquad (2.23)$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric tensor density, with $\epsilon^{01} = 1$, and $R^{(2)}$ is the scalar curvature of the world-sheet metric g, given by

$$R^{(2)} = g_{\mu\nu}R^{\mu\nu} , \quad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} , \quad R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\tau_{\mu\nu}\Gamma^\rho_{\tau\sigma} - \Gamma^\tau_{\mu\sigma}\Gamma^\rho_{\tau\nu} , \qquad (2.24)$$

where $\Gamma^\rho_{\mu\nu}$ are Christoffel symbols (2.6) of the world-sheet metric g. The coupling of the string with the metric tensor is the same as in the case of Polyakov action. The Kalb-Ramond field is analogous to the potential A_μ, and hence this term can be seen as the interacting term. The dilaton is a quantum effect, that is added to preserve the conformal invariance on the quantum level. The conformal invariance on the quantum level results in the space-time equations of motion for the background fields, that to the lowest order in slope parameter $\alpha' = \frac{2\pi}{\kappa}$ are [29]

$$R_{\mu\nu} - \frac{1}{4}B_{\mu\rho\sigma}B_\nu{}^{\rho\sigma} + 2D_\mu D_\nu \Phi = 0 , \qquad (2.25)$$

$$D_\rho B^\rho{}_{\mu\nu} - 2D_\rho \Phi B^\rho{}_{\mu\nu} = 0 , \qquad (2.26)$$

$$4(D\Phi)^2 - 4D_\mu D^\mu \Phi + \frac{1}{12}B_{\mu\nu\rho}B^{\mu\nu\rho} - R = 0 , \qquad (2.27)$$

where $B_{\mu\nu\rho}$ is the Kalb-Ramond field strength, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} , \qquad (2.28)$$

R and $R_{\mu\nu}$ are respectively the Ricci scalar and Ricci tensor (2.24) of the space-time metric $G_{\mu\nu}$, and by D_μ we have marked the covariant derivative, which acts on a vector field V_μ by

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu}V_\sigma , \qquad (2.29)$$

[1] It is possible to define non-critical bosonic string theory in a space-time with dimensions $D < 26$, provided that an appropriate Liouville term is added to the action. This results in a Liouville field theory that is not linear and does not possess the Weyl invariance but is classically integrable. For reviews of the Liouville field theory, see [27, 28].

which is easily generalized to the terms appearing in the equations of motions

$$
\begin{aligned}
D_\mu D_\nu \Phi &= \partial_\mu D_\nu \Phi - \Gamma^\sigma_{\mu\nu} D_\sigma \Phi \,, \\
D_\mu B^\nu{}_{\rho\sigma} &= \partial_\mu B^\nu{}_{\rho\sigma} + \Gamma^\nu_{\mu\tau} B^\tau{}_{\rho\sigma} - \Gamma^\tau_{\mu\rho} B^\nu{}_{\tau\sigma} - \Gamma^\tau_{\mu\sigma} B^\nu{}_{\rho\tau} \,.
\end{aligned}
\tag{2.30}
$$

The conformal symmetry allows us to chose the conformal gauge $g_{\alpha\beta} = e^{2\varphi}\eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is a flat Minkowski metric. Furthermore, if we do not take into account quantum effects, the dilaton field can be taken to be zero, in which case the action simplifies to

$$
S = \int_\Sigma d^2\xi \mathcal{L} = \kappa \int_\Sigma d^2\xi \Big(\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu} \Big) \partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{2.31}
$$

which in the light-cone coordinates becomes

$$
S = \kappa \int d\xi^2 \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \,, \tag{2.32}
$$

where $\Pi_{\pm\mu\nu}$ fields are given by

$$
\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu} \,. \tag{2.33}
$$

The action in the form (2.31) will mostly be used in this thesis. The symmetries of this action will be analyzed in detail.

2.4 Canonical Hamiltonian

We finish this chapter with the derivation of the canonical Hamiltonian for string σ-model. The canonical momenta corresponding to the coordinate x^μ are obtained from variation of the Lagrangian (2.31) with respect to coordinate time derivative

$$
\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu} \dot{x}^\nu - 2\kappa B_{\mu\nu} x'^\nu \,. \tag{2.34}
$$

The canonical Hamiltonian is the Legendre transformation of the Lagrangian

$$
\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa} \pi_\mu (G^{-1})^{\mu\nu} \pi_\nu + \frac{\kappa}{2} x'^\mu G^E_{\mu\nu} x'^\nu - 2x'^\mu B_{\mu\rho} (G^{-1})^{\rho\nu} \pi_\nu \,, \tag{2.35}
$$

where G_E is the effective metric, given by

$$
G_E^{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} \,. \tag{2.36}
$$

The effective metric is the open string metric. From the symmetric properties of the background fields B and G, one easily shows that the effective metric is symmetric.

Often we will find convenient to express the Hamiltonian in matrix notation

$$\mathcal{H}_C = \frac{1}{2\kappa}(X^T)^M H_{MN} X^N \,, \tag{2.37}$$

where H_{MN} is the generalized metric, given by

$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & 2(BG^{-1})_\mu{}^\nu \\ -2(G^{-1}B)^\mu{}_\nu & (G^{-1})^{\mu\nu} \end{pmatrix} \,, \tag{2.38}$$

and

$$X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix} \,. \tag{2.39}$$

The index M can take values from 0 to $2D-1$, and includes both covariant and contravariant indices.

Chapter 3

T-duality

Dualities are well known to appear in physics. There is a unique form of duality that appears in string theory and relates theories formulated in different geometries or topologies, called T-duality. In this chapter, we will start with the presentation of the first emergence of T-duality, in the case of a string moving in a background with one dimension compactified to a circle. Next, we will show how to obtain the T-dual theory from the initial one by means of the Buscher procedure. Lastly, we will comment on how the procedure can be generalized and what are the other important features of T-duality.

3.1 First appearance of T-duality

The most well-known example where T-duality emerges is a closed bosonic string in the space-time with one dimension compactified to a circle of radius R. In that case, the space-time is the tensor product of Minkowski space-time and a circle $\mathbb{R}^{1,24} \times S^1$. The compactification on a circle has a couple of peculiar consequences. Firstly, the generator of translation by a along the dimension x_{25} is proportional to the factor $e^{ip_{25}a}$. The translation by $2\pi R$ should by design be the identity operator $e^{ip_{25}\cdot 2\pi R} = 1$, from which we obtain

$$p^{25} = \frac{n}{R}, \quad n \in \mathbb{Z}\,, \tag{3.1}$$

where n are integers known as momentum numbers. Secondly, the string can wind around the compact dimension, which we can express by

$$x^{25}(\sigma + 2\pi) = x^{25}(\sigma) + 2\pi m R\,, \quad m \in \mathbb{Z}\,, \tag{3.2}$$

where m is the winding number, equals the number of times the string winds around the compact dimension. The mass spectrum of particles can be obtained in the form

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{l_s^4}\,. \tag{3.3}$$

The spectrum remains invariant under the exchange of winding numbers m and momentum numbers n, provided that one makes the transformation $R \leftrightarrow \frac{l_s^2}{R}$.

The closed string moving on a dimension compactified on a radius R has the indistinguishable physics from the string moving on a dimension compactified on a radius $\frac{l_s^2}{R}$. Though two strings would be described by different actions, the observable quantities would be the same. Moreover, the self T-dual radius $R = l_s$ can be seen as the critical, or minimal radius. All theories with a dimension compactified on radii lower than the critical one are in fact T-dual to theories with a dimension compactified on a larger one.

In the case of a large radius $R \to \infty$, winding modes become very heavy and hence require a lot of energy to be created. As such, they become irrelevant for the dynamics of a string. The momentum modes become very light, and the differences between two modes $\frac{l_s^2}{R}$ becomes very small, effectively meaning that momenta are continuous. The large radius limit $R \to \infty$ describes a string with no winding and continuous momenta along that dimension, which is equivalent to that dimension being effectively non-compactified. In its corresponding T-dual radius $R \to 0$, we have the opposite case. The momentum modes are very heavy, but the winding modes are very light and basically make the continuum.

3.2 Buscher procedure

The Buscher procedure is the first formal method of obtaining T-dual theory from the closed string σ-model. The procedure requires a shift of coordinate

$$\delta x^\mu = \lambda^\mu = const \;, \tag{3.4}$$

to be a global symmetry of the action (2.31), which corresponds to the case of constant background fields

$$B_{\mu\nu} = const \,, \quad G_{\mu\nu} = const \,. \tag{3.5}$$

The first step in the procedure is a localization of the global symmetry. The partial derivative is replaced with the covariant derivative

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v_\alpha^\mu \,, \tag{3.6}$$

where v_α^μ are gauge fields. We will require the covariant derivatives to be gauge invariant

$$\delta D_\alpha x^\mu = 0 \,, \tag{3.7}$$

from which we can easily read the transformation laws for the gauge fields

$$\delta v_\alpha^\mu = -\partial_\alpha \lambda^\mu \,. \tag{3.8}$$

Since the gauge fields are not physical, we will demand that their field strength is zero

$$F^{\mu}_{\alpha\beta} = \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha = 0\,. \tag{3.9}$$

This condition can be assured by adding the appropriate term with the Lagrangian multiplier y_μ to the action (2.31). The gauge action invariant under the localized symmetries is then given by

$$S_{loc}(x,y,v) = \kappa \int d^2\xi \Big[(\frac{\eta^{\alpha\beta}}{2} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}) D_\alpha x^\mu D_\beta x^\nu + \frac{1}{2} y_\mu \epsilon^{\alpha\beta} F^\mu_{\alpha\beta} \Big]\,. \tag{3.10}$$

In the second step, we will fix the gauge by demanding $x^\mu(\xi) = x^\mu(\xi_0)$, so the gauge fixed action is

$$S_{fix}(y,v) = \kappa \int d^2\xi \Big[(\frac{\eta^{\alpha\beta}}{2} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}) v^\mu_\alpha v^\nu_\beta + \frac{1}{2} y_\mu \epsilon^{\alpha\beta} F^\mu_{\alpha\beta} \Big]\,, \tag{3.11}$$

or in the light-cone coordinates (2.19)

$$S_{fix}(y,v) = \kappa \int d^2\xi \Big[v^\mu_+ \Pi_{+\mu\nu} v^\nu_- + \frac{1}{2} y_\mu (\partial_+ v^\mu_- - \partial_- v^\mu_+) \Big]\,. \tag{3.12}$$

Equations of motions can be obtained from the variation principle. By varying with respect to the Lagrange multiplier y_μ, one obtains the condition (3.9), as required. Its solution is

$$v^\mu_\pm = \partial_\pm x^\mu\,, \tag{3.13}$$

which when substituted in the gauge fixed action (3.12) gives rise to the initial action (2.32). On the other hand, the variation with respect to the gauge field produces another set of equations of motions

$$\partial_\pm y_\mu + 2\Pi_{\mp\mu\nu} v^\nu_\pm = 0\,. \tag{3.14}$$

To find their solution, we introduce another set of fields by

$$\Theta^{\mu\nu}_\pm = \theta^{\mu\nu} \mp \frac{1}{\kappa} (G^{-1}_E)^{\mu\nu}\,, \tag{3.15}$$

where $\theta^{\mu\nu}$ is the non-commutativity parameter, given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa} (G^{-1}_E B G^{-1})^{\mu\nu}\,, \tag{3.16}$$

and G^{-1}_E is the inverse of the effective metric (2.36). The non-commutativity parameter appears in non-commutative relations on the open string endpoints, in the presence of non-zero Kalb-Ramond field [30]. It is an antisymmetric tensor, while the inverse of the effective metric is symmetric, so one easily proves that

$$\Theta^{\mu\nu}_\pm = -\Theta^{\nu\mu}_\mp\,, \tag{3.17}$$

and moreover that

$$\Pi_{\pm\mu\rho}\Theta^{\rho\nu}_{\mp} = \frac{1}{2\kappa}\delta^{\nu}_{\mu} \, . \tag{3.18}$$

We can multiply the equation of motion (3.14) with $\Theta_{\pm}$ and obtain its solution

$$v^{\mu}_{\pm} = -\kappa\Theta^{\mu\nu}_{\pm}\partial_{\pm}y_{\nu} \, . \tag{3.19}$$

When substituted (3.19) in the gauge fixed action, the T-dual action is obtained

$$^{\star}S(y) = \int d\xi^{2\star}\mathcal{L} = \kappa\int d\xi^{2}\partial_{+}y_{\mu}{}^{\star}\Pi^{\mu\nu}_{+}\partial_{-}y_{\nu} \, , \quad {}^{\star}\Pi^{\mu\nu}_{+} = \frac{\kappa}{2}\Theta^{\mu\nu}_{-} \, , \tag{3.20}$$

where y_{μ} is the T-dual coordinate, and the T-dual fields are given by

$$^{\star}G^{\mu\nu} = (G^{-1}_{E})^{\mu\nu} \, , \quad {}^{\star}B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu} \, . \tag{3.21}$$

By comparing (3.13) and (3.18), one obtains the T-duality relations between the coordinates

$$\partial_{\pm}x^{\mu} \simeq -\kappa\Theta^{\mu\nu}_{\pm}\partial_{\pm}y_{\nu} \, , \tag{3.22}$$

or

$$\begin{aligned} \dot{x}^{\mu} &\simeq -\kappa\theta^{\mu\nu}\dot{y}_{\nu} + (G^{-1}_{E})^{\mu\nu}y'_{\nu} \, , \\ x'^{\mu} &\simeq (G^{-1}_{E})^{\mu\nu}\dot{y}_{\nu} - \kappa\theta^{\mu\nu}y'_{\nu} \, . \end{aligned} \tag{3.23}$$

When these transformations are applied to the coordinates in the initial Lagrangian, the T-dual Lagrangian is obtained, i.e.

$$\begin{aligned} \kappa\partial_{+}x^{\mu}\Pi_{+\mu\nu}\partial_{-}x^{\nu} &\simeq \kappa^{3}\Theta^{\mu\rho}_{+}\partial_{+}y_{\rho}\Pi_{+\mu\nu}\Theta^{\nu\sigma}_{-}\partial_{-}y_{\sigma} = -\kappa^{3}\partial_{+}y_{\rho}\Theta^{\rho\mu}_{-}\Pi_{+\mu\nu}\Theta^{\nu\sigma}_{-}\partial_{-}y_{\sigma} \\ &= -\frac{\kappa^{2}}{2}\partial_{+}y_{\rho}\Theta^{\rho\sigma}_{-}\partial_{-}y_{\sigma} \, . \end{aligned} \tag{3.24}$$

The Buscher procedure can be applied to the T-dual action (3.20) as well. We will not go into details, as the procedure is exactly the same as when it is applied to the initial action. The T-duality transformation laws on the T-dual coordinates are given by [16]

$$\partial_{\pm}y_{\mu} \simeq -2\Pi_{\mp\mu\nu}\partial_{\pm}x^{\nu} \, . \tag{3.25}$$

When these relations are applied to the T-dual action (3.20), the initial action is obtained (2.32).

The T-duality relations are as easily expressed in Hamiltonian formalism. The variation of the T-dual Lagrangian with respect to the T-dual coordinate τ-derivative is equal to the T-dual canonical momentum

$$^{\star}\pi^{\mu} = \frac{\partial\, {}^{\star}\mathcal{L}}{\partial\dot{y}_{\mu}} = \kappa(G^{-1}_{E})^{\mu\nu}\dot{y}_{\nu} - \kappa^{2}\theta^{\mu\nu}y'_{\nu} \, . \tag{3.26}$$

Therefore, in terms of the canonical variables, T-dual relations (3.23) are

$$\pi_\mu \simeq \kappa y'_\mu \,, \quad \kappa x'^\mu \simeq {}^\star\pi^\mu \,. \tag{3.27}$$

The T-duality transforms momenta into σ-derivatives of the T-dual coordinates, and vice versa. To give a further interpretation of these relations, we notice that the integrals of the canonical momenta produce the momentum numbers P^μ, and the integrals of the σ-derivatives of the coordinates along compact dimension produce the winding numbers W^μ

$$P^\mu = \int d\sigma \pi_\mu \,, \quad W^\mu = \int d\sigma \kappa x'^\mu \,. \tag{3.28}$$

The Buscher procedure demonstrates that the winding numbers of the initial theory are the momenta in its T-dual theory, and vice versa.

3.3 Beyond Buscher procedure

Due to its requirement for global shift symmetry, the Buscher procedure cannot be applied in the majority of cases. Let us consider a so-called weakly curved background, characterized by the constant metric field $G_{\mu\nu} = const$ and the Kalb-Ramond field only linearly dependent on coordinate $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho \,, \quad b_{\mu\nu} = const$. If the field strength $B_{\mu\nu\rho}$ is taken to be infinitesimal and its higher orders are neglected, it is straightforward to demonstrate the fields in weakly curved background satisfy equations of motion (2.25) - (2.27). In this case, it is possible to construct the adapted Buscher procedure [31]. One introduces the gauge fields v^μ_α and replaces the partial derivatives with the covariant ones. This is insufficient, because the Kalb-Ramond field B depends on x^μ, and the coordinate x^μ itself is not locally gauge invariant. This obstacle can be overcome by replacing the coordinate with the line integral

$$V^\mu = \int_P d\xi^\alpha v^\mu_\alpha = \int_P (d\xi^+ v^\mu_+ + d\xi^- v^\mu_-) \,, \tag{3.29}$$

where the integration is done along the path from the initial point $\xi^\alpha_0(\tau_0, \sigma_0)$ to the final one $\xi^\alpha(\tau, \sigma)$. With this change in mind, one can follow Buscher's ideas. On the equations of motions related to the Lagrangian multiplier, the initial theory is obtained. On the equations of motions for the gauge fields v^μ_α, the T-dual theory is obtained. The background fields in the T-dual theory depend on the line integral of T-dual coordinate y_μ. The line integral is a non-local object, and we say that this theory is non-geometric. The situation becomes even more complicated when other backgrounds are included, where the metric also depends on coordinates [32].

3.4 T-duality in superstring theories

Even though this thesis is primarily focused on bosonic strings, we are going to briefly touch on the importance of T-duality in superstring theories. The bosonic string theory is incomplete because it does not include fermions. They can be added to the theory with the introduction of supersymmetry. We can require that the theory is either supersymmetric on the world-sheet, in which case we obtain the Ramond-Neveu-Schwarz (RNS) strings, or that the theory is supersymmetric in the ten-dimensional Minkowski space-time, in which case we obtain the Green-Schwarz (GS) strings.

The RNS formalism [33, 34] relies on adding the standard Dirac action for D massless fermions. The action is invariant under the $N = 2$ supersymmetry group. The left and right moving ground fermionic states can be chosen to have the same or the opposite chirality. The former case corresponds to the type IIA superstring theory, while the latter case corresponds to the type IIB superstring theory. These two superstring theories contain only closed strings.

The supersymmetric theory that contains both open and closed strings can be constructed with the help of GS mechanism [35, 36, 37], where the action is constructed by requiring the manifest Poincaré supersymmetry. This action can be only invariant under $N = 1$ supersymmetry, in which case we obtain type I superstring theory.

Heterotic string theories [38] can be constructed by combining the left movers of the 26-dimensional bosonic string theory, with the right movers of the ten-dimensional superstring theory. The spectrum contains Yang-Mills multiplets that are either based on $SO(32)$ or $E_8 \times E_8$ gauge symmetry.

The fact that we are able to construct five different but self-consistent superstring theories was initially seen as a weakness of string theory since there was no obvious choice of which one should be preferable. However, these theories are mutually related by dualities. Specifically, T-duality connects IIA and IIB superstring theories [39, 40], and also two heterotic string theories [41, 42, 43].

Apart from T-duality, different superstring theories can also be related via S-duality. It is a duality between theories that have different coupling constants. Type I superstring theory with a coupling constant g is S-dual to the $SO(32)$ heterotic string theory with coupling constant $\frac{1}{g}$ [44]. Moreover, type IIB superstring theory is invariant under S-duality.

In 1995, Edward Witten suggested that all superstring theories could be related with dualities to an eleven-dimensional M-theory. The M-theory has not been formulated on all orders of perturbations, but its effective low-energy action is an eleven-dimensional theory of supergravity. Type IIA superstring theory in strong coupling is equivalent to an eleven-dimensional supergravity theory with one dimension compactified on T^1 [44]. Similarly, $E_8 \times E_8$ heterotic string theory with strong coupling constant g becomes M-theory compactified to a Z_2 orbifold of the circle [45, 46]. These observations make strong arguments for the existence of the M-theory which is connected with all superstring theories by a web of dualities.

So far, there is no universal description of T-duality that can be applied to an arbitrary string field configuration. The intricate geometric and non-geometric spaces that can be related via T-duality necessitate a novel mathematical framework to be able to accommodate them. In the next part, we will introduce the reader to the basic elements of generalized geometry, which appears to come with a suitable apparatus for describing T-duality and other relevant string phenomena.

Part II

Generalized geometry

Chapter 4

Differential geometry

This chapter begins with definitions of fundamental geometric terms that will be used throughout the thesis. We introduce vectors, the Lie derivative of a vector field, and define the Lie bracket. Next, we define differential forms, together with the exterior derivative and the interior product. Finally, we extend the definition of Lie bracket to multi-vectors by introducing the Schouten-Nijenhuis bracket.

4.1 Tangent and cotangent bundle

We start with the definition of a manifold:

Definition 1 (Manifold)
A n-dimensional smooth manifold is a Hausdorff[1] *topological space $\mathcal{M}$ such that:*

- *For each point $p \in \mathcal{M}$ there is an open neighborhood U_α that is homeomorphic to $\mathbb{R}^n$, i.e. there is a smooth map $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. The ordered pairs U_α, ϕ_α are called charts, and the collection of charts covering all topological space is called atlas.*
- *For each two non-disjoint neighborhoods U_α and U_β, transition maps on their intersections $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are smooth maps.*

This means that locally n-dimensional manifold resembles the n-dimensional Euclidean space, and that in each chart a local coordinate system is defined. The resemblance with $\mathbb{R}^n$ manifests in the ability to use the calculus techniques on manifolds.

[1]Hausdorff topological space is a topological space where for any two distinct points, there exists neighborhoods of each which are disjoint from each other.

Definition 2 (Tangent vector)

Let $\mathcal{M}$ be a smooth n-dimensional manifold. Given a point p on it, let $\mathcal{F}_p\mathcal{M}$ be the family of real valued smooth functions on $\mathcal{M}$. A function $\xi : \mathcal{F}_p\mathcal{M} \to \mathbb{R}$ is called the tangent vector ξ at $p \in \mathcal{M}$ if it satisfies:

- *ξ is linear - $\xi(af + bg) = a\xi(f) + b\xi(g)$, for $f,\ g \in \mathcal{F}_p\mathcal{M}$, and for $a,\ b \in \mathbb{R}$,*
- *ξ satisfies Leibniz property - $\xi(fg) = \xi(f)g + f\xi(g)$, for $f,\ g \in \mathcal{F}_p\mathcal{M}$.*

If we chose a chart at p with coordinates x^μ, the vector field in point p can be represented in the basis of partial derivatives $\xi = \xi^\mu \partial_\mu$. If there is no global covering of the manifold, i.e. if it cannot be covered by a single chart, then it is not possible to define partial derivatives globally. The set of all tangent vectors at a point p forms a vector space.

Definition 3 (Tangent space)

The set $T_p\mathcal{M}$ of all tangent vectors through a point p is called the tangent space of $\mathcal{M}$ at p.

Intuitively, we can associate a n-dimensional Euclidean vector space to each point of a manifold. This is by no means sufficient to describe all physical phenomena. For example, there are internal degrees of freedom, that are associated to each point on the space-time, and as such we would like to have generalizations of manifolds, such that to each point on a manifold we can attribute another manifold. To achieve this, let us define the fiber bundle.

Definition 4 (Fiber bundle)

A bundle is a triple $\Big(V,\ \pi,\ \mathcal{M}\Big)$ consisting of a base manifold $\mathcal{M}$, a total manifold V and a surjective map $\pi : V \to \mathcal{M}$ called projection. The inverse image $\pi^{-1}(p)$ is the fiber over p, for $p \in \mathcal{M}$. If fibers over all points on the base manifold are homeomorphic to a space F, the triple $\Big(V,\ \pi,\ \mathcal{M}\Big)$ is said to be a fiber bundle, with F being a fiber.

The physical fields are usually represented as functions that depend on specific points on manifolds, and appear as sections.

Definition 5 (Section)

A section of a bundle $\Big(V,\ \pi,\ \mathcal{M}\Big)$ is a map $\sigma : \mathcal{M} \to V$ such that the image of each point $p \in \mathcal{M}$ lies in the fiber $\pi^{-1}(p)$ over p, i.e. $\pi \circ \sigma =$ Id, where Id is an identity operator on the base manifold.

The first example of a fiber bundle can simply be obtained as the disjoint union of the tangent spaces of a manifold $\mathcal{M}$. This way, we obtain the tangent bundle, which we define below.

Definition 6 (Tangent bundle)

The tangent bundle is a triple $\left(T\mathcal{M},\ \pi,\ \mathcal{M}\right)$*, where* $T\mathcal{M}$ *is the disjoint union of the tangent spaces of a base manifold* $\mathcal{M}$*, and the projection* π *is trivial projection* $\pi : T_p\mathcal{M} \to p$*.*

The fibers of the tangent bundle are the tangent spaces $T_p\mathcal{M}$. Its section is some function that will take some point p on the manifold as a domain and map it to the fiber $T_p\mathcal{M}$. Therefore, we conclude that vector fields are elements of the smooth section of the tangent bundle.

The tangent bundle is also a smooth manifold, so we can define the higher order tangent bundles by $T^n\mathcal{M} = T(T^{n-1}\mathcal{M})$. Their sections' elements are antisymmetric multi-vectors.

We also define the cotangent space and cotangent bundle:

Definition 7 (Cotangent bundle)

The cotangent space $T_p^\star\mathcal{M}$ *to a smooth manifold* $\mathcal{M}$ *at the point* p *is the dual space of the tangent space* $T_p\mathcal{M}$*. The disjoint union of cotangent spaces to the manifold is the cotangent bundle* $T^\star\mathcal{M}$*.*

The elements of the smooth section of the cotangent bundle are differential 1-forms. In some local chart, 1-forms can be written in the basis of coordinate differentials, i.e. $\lambda = \lambda_\mu dx^\mu$.

The cotangent bundle is also a manifold, so higher order of cotangent bundles are defined as the cotangent bundles of the cotangent bundles, i.e. $\bigwedge^p T^\star\mathcal{M} = T^\star \bigwedge^{p-1} T^\star\mathcal{M}$. The elements of their sections are differential p-forms ω - totally antisymmetric tensors of type $(0, p)$, which in some coordinate basis can be expressed by

$$\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \,, \tag{4.1}$$

where $\wedge$ denotes the wedge product $\wedge : \bigwedge^p T^\star\mathcal{M} \times \bigwedge^q T^\star\mathcal{M} \to \bigwedge^{p+q} T^\star\mathcal{M}$, which is totally antisymmetric, e.g. $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$.

4.2 Lie derivative

Lie derivative represents the coordinate invariant evaluation of the change of a tensor field along the flow defined by a vector field. There are multiple equivalent ways to define a Lie derivative. We present the so-called classical definition:

Definition 8 (Lie derivative)

Let T *be a tensor field of type (p, q) (i.e. contravariant of order p and covariant of order q) defined over a manifold* $\mathcal{M}$*. The Lie derivative of the tensor field T with respect to the vector field* ξ *is another tensor field* $\mathcal{L}_\xi T$ *of type (p, q) with components*

$$\mathcal{L}_\xi T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} = \xi^\rho \partial_\rho T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} - \sum_{i=0}^{p} T^{\mu_1\ldots\rho\hat{\mu}_i\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q}\partial_\rho\xi^{\mu_i} + \sum_{i=1}^{q} T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\rho\hat{\nu}_i\ldots\nu_q}\partial_{\nu_i}\xi^\rho \,, \tag{4.2}$$

where $\hat{\nu}_i$ denotes omission of such index, e.g. $T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\ldots\rho\hat{\nu}_q}\partial_{\nu_q}\xi^\rho = T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\rho}\partial_{\nu_q}\xi^\rho$

We will now provide explicit expressions for the action of the Lie derivative on the tensors that we will encounter most frequently. Lie derivative of a function is defined as the directional derivative of a function

$$\mathcal{L}_\xi f = \xi^\mu \partial_\mu f \,. \tag{4.3}$$

The Lie derivative of a vector field produces another vector field, that is known as the Lie bracket, defined by

$$[\xi_1, \xi_2]_L f = (\mathcal{L}_{\xi_1}\xi_2) f = \mathcal{L}_{\xi_1}(\mathcal{L}_{\xi_2} f) - \mathcal{L}_{\xi_2}(\mathcal{L}_{\xi_1} f) \,, \tag{4.4}$$

or in local coordinate basis by

$$\Big([\xi_1, \xi_2]_L\Big)^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu \,. \tag{4.5}$$

The Lie bracket satisfies the Leibniz rule

$$[\xi_1, f\xi_2]_L = f[\xi_1, \xi_2]_L + (\mathcal{L}_{\xi_1} f)\, \xi_2 \,, \tag{4.6}$$

and the Jacobi identity

$$[\xi_1, [\xi_2, \xi_3]_L]_L + [\xi_2, [\xi_3, \xi_1]_L]_L + [\xi_3, [\xi_1, \xi_2]_L]_L = 0 \,. \tag{4.7}$$

We can use the Lie derivative properties in order to obtain the action of Lie derivative on the 1-forms. There is a way to obtain a scalar from a vector ξ with a 1-form λ using the contraction defined by $\lambda(\xi) = \lambda_\mu \xi^\mu$. From the fact that Lie derivative satisfies the Leibniz rule, we write

$$\mathcal{L}_{\xi_1}(\lambda(\xi_2)) \;=\; (\mathcal{L}_{\xi_1}\lambda)\xi_2 + \lambda(\mathcal{L}_{\xi_1}\xi_2) = (\mathcal{L}_{\xi_1}\lambda)_\mu \xi_2^\mu + \lambda_\mu\Big(\xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu\Big) \,. \tag{4.8}$$

On the other hand, $\lambda(\xi_2)$ is a scalar, and using (4.3) we obtain

$$\mathcal{L}_{\xi_1}(\lambda(\xi_2)) = \xi_1^\mu \partial_\mu(\lambda_\nu \xi_2^\nu) = \xi_1^\mu \partial_\mu \lambda_\nu \xi_2^\nu + \xi_1^\mu \lambda_\nu \partial_\mu \xi_2^\nu \,. \tag{4.9}$$

By comparing the relations (4.8) and (4.9), we obtain the action of the Lie derivative on 1-form expressed in some coordinate basis by

$$\Big(\mathcal{L}_{\xi_1}\lambda\Big)_\mu = \xi_1^\nu(\partial_\nu \lambda_\mu - \partial_\mu \lambda_\nu) \,. \tag{4.10}$$

4.3 Exterior algebra of differential forms

Differential p-forms (4.1) are part of the smooth section of $\bigwedge^p T\mathcal{M}$. The antisymmetric wedge product $\wedge$ defines exterior algebra between differential forms, equipped with natural grading related to the degree of differential forms. It is often convenient working with the exterior derivative and interior product, which we define below.

Definition 9 (Exterior derivative)

The exterior derivative of a p-form λ is a $p+1$ -form $d\lambda$, such that

$$\begin{aligned} d\lambda(\xi_0, ..., \xi_p) \;\; = \;\; & \sum_{i=0}^{p} (-1)^i \mathcal{L}_{\xi_i} \Big(\lambda(\xi_0, ..., \hat{\xi}_i, ..., \xi_p) \Big) \\ & + \sum_{i<j} (-1)^{i+j} \lambda([\xi_i, \xi_j]_L, \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_p) \,, \end{aligned} \tag{4.11}$$

where $d\lambda(\xi_0, ..., \xi_p)$ stands for the contraction of a $p+1$ -form $d\lambda$ with $p+1$ vectors $\xi_0 ... \xi_p$, and $\hat{\xi}_i$ denotes the omission of ξ_i in such contractions.

The exterior derivative extends the notion of the differential of the function to differential forms of degree p. It is the antiderivation of degree 1 on the graded algebra of differential forms $\bigwedge^p T^\star\mathcal{M} \rightarrow \bigwedge^{p+1} T^\star\mathcal{M}$ that is also nilpotent, i.e.

$$d(d\lambda) = 0 \,, \tag{4.12}$$

for any p-form λ, and it satisfies the graded Leibniz identity

$$d(\lambda_1 \wedge \lambda_2) = d\lambda_1 \wedge \lambda_2 + (-1)^p \lambda_1 \wedge d\lambda_2 \,, \tag{4.13}$$

where $\lambda_1,\ \lambda_2$ are a p-form and a q-form, respectively.

Definition 10 (Interior product)

The interior product $i_\xi : \bigwedge^p T^\star\mathcal{M} \rightarrow \bigwedge^{p-1} T^\star\mathcal{M}$ is defined to be the contraction of a differential form with a vector field ξ by

$$(i_\xi \lambda)(\xi_1, \xi_2, ..., \xi_{p-1}) = \lambda(\xi, \xi_1, \xi_2, ..., \xi_{p-1}) \,. \tag{4.14}$$

The interior product reduces the degree of a differential form by one, and also satisfies the graded Leibniz identity

$$i_\xi(\lambda_1 \wedge \lambda_2) = i_\xi \lambda_1 \wedge \lambda_2 + (-1)^p \lambda_1 \wedge i_\xi \lambda_2 \,, \tag{4.15}$$

for any p-form λ_1, and q-form λ_2, so it is an antiderivation of degree -1 on the graded algebra of differential forms.

The Lie derivative on a function (4.3) can be written in terms of interior product by

$$\mathcal{L}_\xi f = i_\xi df \,, \tag{4.16}$$

while the Lie derivative of a 1-form (4.10) can be rewritten in coordinate independent form, also known as Cartan formula, by

$$\mathcal{L}_\xi \lambda = i_\xi d\lambda + di_\xi \lambda \,. \tag{4.17}$$

The Cartan formula stands for any p-form λ. Combining Cartan formula with Leibniz rule, we can obtain the useful identity for Lie derivative

$$\mathcal{L}_{(f\xi)}\lambda = f\mathcal{L}_\xi \lambda + df i_\xi \lambda \,, \tag{4.18}$$

where f is a smooth function.

For two vector fields ξ_1 and ξ_2, the interior product satisfies

$$i_{\xi_1} i_{\xi_2} \lambda = -i_{\xi_2} i_{\xi_1} \lambda \,, \tag{4.19}$$

and

$$i_{[\xi_1,\xi_2]_L} = \mathcal{L}_{\xi_1} i_{\xi_2} - i_{\xi_2}\mathcal{L}_{\xi_1} \,. \tag{4.20}$$

4.4 Schouten-Nijenhuis bracket

The Schouten-Nijenhuis bracket [47, 48, 49] is a bracket that extends the notion of the Lie bracket to the space of multi-vectors. Formally, multi-vectors and Schouten-Nijenhuis bracket constitute a Gerstenhaber algebra, which is a graded-commutative algebra with a Lie bracket of degree -1 satisfying the Poisson identity.

Let $\theta_1 \in \Gamma(T \bigwedge^p \mathcal{M})$ and $\theta_2 \in \Gamma(T \bigwedge^q \mathcal{M})$ be multi-vectors of order p and q respectively, and $0 \leq p, q \leq dim(\mathcal{M})$. Suppose that in some local coordinate basis they are given by

$$\theta_1 = \frac{1}{p!}\theta_1^{\mu_1,\ldots\mu_p}\partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_p} \,, \quad \theta_2 = \frac{1}{q!}\theta_2^{\nu_1,\ldots\nu_q}\partial_{\nu_1} \wedge \ldots \wedge \partial_{\nu_q} \,. \tag{4.21}$$

The Schouten-Nijenhuis bracket of θ_1 and θ_2 is the function $[,]_S : T \bigwedge^p \mathcal{M} \times T \bigwedge^q \mathcal{M} \to T \bigwedge^{p+q-1} \mathcal{M}$, given by

$$\begin{aligned} [\theta_1, \theta_2]_S &= \frac{1}{(p+q-1)!}[\theta_1,\theta_2]_S^{\mu_1\ldots\mu_{p+q-1}}\partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_{p+q-1}} \,, \\ [\theta_1,\theta_2]_S^{\mu_1\ldots\mu_{p+q-1}} &= \frac{1}{(p-1)!q!}\epsilon^{\mu_1\ldots\mu_{p+q-1}}_{\nu_1\ldots\nu_{p-1}\rho_1\ldots\rho_q}\theta_1^{\sigma\nu_1\ldots\nu_{p-1}}\partial_\sigma \theta_2^{\rho_1\ldots\rho_q} \\ &\quad + \frac{(-1)^p}{p!(q-1)!}\epsilon^{\mu_1\ldots\mu_{p+q-1}}_{\nu_1\ldots\nu_p\rho_1\ldots\rho_{q-1}}\theta_2^{\sigma\rho_1\ldots\rho_{q-1}}\partial_\sigma\theta_1^{\nu_1\ldots\nu_p} \,, \end{aligned} \tag{4.22}$$

where the antisymmetric Levi Civita symbol is defined by

$$\epsilon^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_p} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \ldots & \delta^{\mu_1}_{\nu_p} \\ . & & . \\ . & & . \\ . & & . \\ \delta^{\mu_p}_{\nu_1} & \ldots & \delta^{\mu_p}_{\nu_p} \end{vmatrix} . \tag{4.23}$$

To get a more grasp into definition of the Schouten-Nijenhuis bracket, let us consider some simple examples. Firstly, for two vector fields $\xi_1 = \xi_1^\mu \partial_\mu$ and $\xi_2^\nu \partial_\nu$, we have $p = q = 1$, and Levi Civita symbol (4.23) becomes just the trivial Kroneker delta $\epsilon^\mu_\nu = \delta^\mu_\nu$. The second expression in (4.22) becomes

$$[\xi_1, \xi_2]^\mu_S = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu \,, \tag{4.24}$$

which is just the expression for Lie bracket. Another case that we will consider in the thesis is of a Schouten-Nijenhuis bracket of the bi-vector $\theta = \frac{1}{2}\theta^{\mu\nu}\partial_\mu \wedge \partial_\nu$ with itself. Its expression is given by

$$[\theta, \theta]_S|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} \theta^{\alpha\sigma} \partial_\sigma \theta^{\beta\gamma} \,, \tag{4.25}$$

and

$$\epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix} \,, \tag{4.26}$$

resulting in

$$[\theta, \theta]_S|^{\mu\nu\rho} = \theta^{\mu\sigma} \partial_\sigma \theta^{\nu\rho} + \theta^{\nu\sigma} \partial_\sigma \theta^{\rho\mu} + \theta^{\rho\sigma} \partial_\sigma \theta^{\mu\nu} \,. \tag{4.27}$$

It turns out that this coincides with the expression for the string R-flux. The bi-vector θ such that its Schouten-Nijenhuis bracket gives zero is a Poisson bi-vector, which can be used to define Poisson manifolds (see Appendix [A]).

The Schouten-Nijenhuis bracket is graded-commutative,

$$[\theta_1, \theta_2]_S = -(-1)^{(p-1)(q-1)} [\theta_2, \theta_1]_S \,, \tag{4.28}$$

and it satisfies the graded Jacobi identity

$$(-1)^{(p-1)(r-1)} [\theta_1, [\theta_2, \theta_3]_S]_S + (-1)^{(q-1)(p-1)} [\theta_2, [\theta_3, \theta_1]_S]_S + (-1)^{(r-1)(q-1)} [\theta_3, [\theta_1, \theta_2]_S]_S = 0 \,, \tag{4.29}$$

where $p,\ q,\ r$ are oders of multi-vectors $\theta_1,\ \theta_2,\ \theta_3$, respectively. Moreover, it satisfies the graded Leibniz identity

$$[\theta_1, \theta_2 \wedge \theta_3]_S = [\theta_1, \theta_2]_S \wedge \theta_3 + (-1)^{(p-1)q} \theta_2 \wedge [\theta_1, \theta_3]_S \,. \tag{4.30}$$

In fact, the Schouten-Nijenhuis bracket can be alternatively defined by relations [22]

$$[f, g]_S = 0\,, \quad [\xi, f]_S = \mathcal{L}_\xi(f)\,, \quad [\xi_1, \xi_2]_S = [\xi_1, \xi_2]_L\,, \tag{4.31}$$

where other relations for multi-vectors are obtained by demanding the graded commutativity (4.28) and graded Leibniz identity (4.30).

Chapter 5

Lie algebroid

In general, vector fields can be defined on a smooth section of a vector bundle, that is not necessarily a tangent bundle. This motivates the question of the change of tensors along these vector fields, which Lie algebroids can explain. In this chapter, we provide a definition of Lie algebroid and demonstrate how one can extend the notions of Lie and exterior derivative to some other vector bundles. We also provide a definition of the Koszul bracket, which is the generalization of the Lie bracket to the space of 1-forms. Lastly, we define Lie bialgebroids, which will be useful in introducing the Courant bracket on the generalized tangent bundle.

5.1 Lie algebroid and its corresponding Lie derivative

Definition 11 (Lie algebroid)
Lie algebroid is a triple $\Big(V, [,], \rho\Big)$ *consisting of a vector bundle* V*, the anchor* $\rho : V \to T\mathcal{M}$*, and the skew-symmetric bracket* $[,]$ *on the space of smooth section of* V*, so that the following compatibility conditions are satisfied:*

$$\rho[\xi_1, \xi_2] = [\rho(\xi_1), \rho(\xi_2)]_L \,, \tag{5.1}$$

$$[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + (\mathcal{L}_{\rho(\xi_1)} f)\xi_2 \,, \tag{5.2}$$

$$[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] = 0 \,. \tag{5.3}$$

The Lie algebroid [50, 51] generalizes the notion of the tangent bundle. The first compatibility condition tells us that the anchor ρ is the morphism between vector bundle V and the tangent bundle over a manifold $T\mathcal{M}$ that is compatible with the Lie bracket. This way, we relate the Lie algebroid bracket, defined on some vector bundle to the well-known Lie bracket on the tangent bundle. The remaining two conditions require that the new bracket satisfies the Leibniz rule and Jacobi identity, both satisfied by the Lie bracket.

From the compatibility conditions in the Lie algebroid definition, we see that on this structure we can define a Lie derivative. Its action on functions is defined by

$$\hat{\mathcal{L}}_\xi f = \mathcal{L}_{\rho(\xi)} f \,, \tag{5.4}$$

and on vectors by the Lie algebroid bracket

$$\hat{\mathcal{L}}_{\xi_1} \xi_2 = [\xi_1, \xi_2] \,. \tag{5.5}$$

The second compatibility condition from the definition above ensures that this derivative satisfies the Leibniz property, i.e.

$$\hat{\mathcal{L}}_{\xi_1}(f\xi_2) = f\hat{\mathcal{L}}_{\xi_1}\xi_2 + (\hat{\mathcal{L}}_{\xi_1} f)\xi_2 \,, \tag{5.6}$$

which can be used to obtain its action on dual vectors.

We can also define the exterior derivation by [52]

$$\begin{aligned} \hat{d}\lambda(\xi_0, ..., \xi_p) \;\; &= \;\; \sum_{i=0}^{p} (-1)^i \mathcal{L}_{\rho(\xi_i)} \Big(\lambda(\xi_0, ..., \hat{\xi}_i, ..., \xi_p) \Big) \\ &\quad + \sum_{i<j} (-1)^{i+j} \lambda([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_p) \,, \end{aligned} \tag{5.7}$$

where we effectively in relation (4.11) substituted the Lie derivative with its algebroid counterpart $\hat{\mathcal{L}}_\xi$. This exterior derivative is also nilpotent. The Lie derivative on the dual vectors can be expressed by Cartan formula, i.e.

$$\hat{\mathcal{L}}_\xi \lambda = i_\xi \hat{d}\, \lambda + \hat{d} i_\xi \, \lambda \,. \tag{5.8}$$

There is a myriad of examples of Lie algebroids. The easiest and the most straightforward is the Lie algebroid with the tangent bundle over a manifold $T\mathcal{M}$ as a vector bundle, the Lie bracket $[,]_L$ and the identity operator Id as its anchor. The compatibility conditions from Definition 11 are the well-known characteristics of the Lie bracket (4.6) and (4.7).

5.2 Koszul bracket

We now want to introduce the Lie algebroid on a cotangent bundle over a manifold $T^\star\mathcal{M}$. We will define a morphism from cotangent to tangent bundle by

$$\theta(\lambda_1)\lambda_2 = \theta(\lambda_1, \lambda_2) \,, \quad \Big(\theta(\lambda_1)\Big)^\mu = \lambda_{1\nu}\theta^{\nu\mu} \,, \tag{5.9}$$

where θ is a bi-vector ($\theta^{\mu\nu} = -\theta^{\nu\mu}$) that satisfies

$$[\theta, \theta]_S = 0 \,, \tag{5.10}$$

where $[,]_S$ is the Schouten-Nijenhuis bracket (4.25). The bi-vector that satisfies this condition is called Poisson bi-vector (see Appendix [A]).

We define the Koszul bracket [53] between two 1-forms by

$$[\lambda_1, \lambda_2]_\theta = \mathcal{L}_{\theta(\lambda_1)}\lambda_2 - \mathcal{L}_{\theta(\lambda_2)}\lambda_1 - d(\theta(\lambda_1, \lambda_2)) \,, \tag{5.11}$$

which in some local basis dx^μ has the components

$$\Big([\lambda_1, \lambda_2]_\theta\Big)_\mu = \theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu} - \lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \lambda_{1\rho}\lambda_{2\nu}\partial_\mu\theta^{\rho\nu} \,. \tag{5.12}$$

To show that the structure $\Big(T^\star\mathcal{M}, [,]_\theta, \theta\Big)$ is a Lie algebroid, we need to prove that the three Lie algebroid conditions (5.1)-(5.3) are satisfied. Firstly, in order to show that θ is really a correct anchor, we express the left-hand side of the (5.1) by

$$\Big(\theta([\lambda_1, \lambda_2]_\theta)\Big)_\mu = \theta^{\nu\rho}\theta^{\sigma\mu}(\lambda_{1\nu}\partial_\rho\lambda_{2\sigma} - \lambda_{2\nu}\partial_\rho\lambda_{1\sigma}) + \lambda_{1\rho}\lambda_{2\sigma}\theta^{\nu\mu}\partial_\nu\theta^{\rho\sigma} \,. \tag{5.13}$$

On the other hand, we express the right-hand side of (5.1) by

$$\begin{aligned} ([\theta(\lambda_1), \theta(\lambda_2)]_L)_\mu &= \lambda_{1\nu}\theta^{\nu\rho}\partial_\rho(\lambda_{2\sigma}\theta^{\sigma\mu}) - \lambda_{2\nu}\theta^{\nu\rho}\partial_\rho(\lambda_{1\sigma}\theta^{\sigma\mu}) \\ &= \theta^{\nu\rho}\theta^{\sigma\mu}\Big(\lambda_{1\nu}\partial_\rho\lambda_{2\sigma} - \lambda_{2\nu}\partial_\rho\lambda_{1\sigma}\Big) + \lambda_{1\rho}\lambda_{2\sigma}\Big(\theta^{\rho\nu}\partial_\nu\theta^{\sigma\mu} + \theta^{\sigma\nu}\partial_\nu\theta^{\mu\rho}\Big) \,. \end{aligned} \tag{5.14}$$

Now combining (5.13) and (5.14), we obtain

$$\begin{aligned} \Big(\theta([\lambda_1, \lambda_2]_\theta)\Big)_\mu &= ([\theta(\lambda_1), \theta(\lambda_2)]_L)_\mu - \lambda_{1\rho}\lambda_{2\sigma}\Big(\theta^{\rho\nu}\partial_\nu\theta^{\sigma\mu} + \theta^{\sigma\nu}\partial_\nu\theta^{\mu\rho}\Big) + \lambda_{1\rho}\lambda_{2\sigma}\theta^{\nu\mu}\partial_\nu\theta^{\rho\sigma} \\ &= ([\theta(\lambda_1), \theta(\lambda_2)]_L)_\mu - \lambda_{1\rho}\lambda_{2\sigma}(\theta^{\mu\nu}\partial_\nu\theta^{\rho\sigma} + \theta^{\rho\nu}\partial_\nu\theta^{\sigma\mu} + \theta^{\sigma\nu}\partial_\nu\theta^{\mu\rho}) \\ &= ([\theta(\lambda_1), \theta(\lambda_2)]_L)_\mu \,, \end{aligned} \tag{5.15}$$

where we used (4.22) and the condition (5.10). If the bi-vector is not Poisson, this condition would not be satisfied.

Next, let us show that the Koszul bracket satisfies the Leibniz rule (5.2)

$$\begin{aligned} [\lambda_1, f\lambda_2]_\theta &= \mathcal{L}_{\theta(\lambda_1)}(f\lambda_2) - \mathcal{L}_{f\theta(\lambda_2)}\lambda_1 - d\theta(\lambda_1, f\lambda_2) \\ &= f\mathcal{L}_{\theta(\lambda_1)}\lambda_2 + (\mathcal{L}_{\theta(\lambda_1)}f)\lambda_2 - f\mathcal{L}_{\theta(\lambda_2)}\lambda_1 \\ &\quad -df\theta(\lambda_2, \lambda_1) - df\theta(\lambda_1, \lambda_2) - fd\theta(\lambda_1, \lambda_2) \\ &= f[\lambda_1, \lambda_2]_\theta + \mathcal{L}_{\theta(\lambda_1)}f\lambda_2 \,. \end{aligned} \tag{5.16}$$

We used the fact that the Lie derivative satisfies the Leibniz rule (4.6), its property (4.18), and that the bi-vector is antisymmetric. We did not use the fact that θ is a Poisson bi-vector, so the Koszul bracket satisfies the Leibniz rule regardless of that.

Lastly, the associativity (5.3) is easily proven from the first Lie algebroid condition (5.15) and associativity of Lie derivative (4.7). We write

$$\begin{aligned} [\theta(\lambda_1),[\theta(\lambda_2),\theta(\lambda_3)]_L]_L + cyclic &= [\theta(\lambda_1),\theta([\lambda_2,\lambda_3]_\theta)]_L + cyclic \\ &= \theta([\lambda_1,[\lambda_2,\lambda_3]_\theta]_\theta + cyclic) = 0\,. \end{aligned} \tag{5.17}$$

We showed that the triple $\Big(T^\star\mathcal{M},[,]_\theta,\theta\Big)$ is a Lie algebroid, for a Poisson bi-vector θ. In the case of the bi-vector not being Poisson, the Koszul bracket can still be defined and it will satisfy the Leibniz rule. However, the anchor θ is not algebra homomorphism, and the Jacobi identity does not stand. This structure is then referred to as quasi-Lie algebroid.

For the Lie algebroid associated with the Koszul bracket, we can define its corresponding Lie derivative. On functions, it can be defined from (5.4)

$$\hat{\mathcal{L}}_{\lambda_1} f = \lambda_{1\nu}\theta^{\nu\mu}\partial_\mu f\,, \tag{5.18}$$

while on 1-forms, it acts as the Koszul bracket

$$\hat{\mathcal{L}}_{\lambda_1}\lambda_2 = [\lambda_1,\lambda_2]_\theta\,. \tag{5.19}$$

Therefore, the Koszul bracket is interpreted as the generalization of the Lie bracket on 1-forms.

Since this Lie algebroid is defined on the cotangent bundle, and therefore acts on 1-forms, it defines the exterior derivation on the smooth section of the tangent bundle and higher orders of the tangent bundle. From (5.7), we obtain the action of exterior derivative on functions and vectors

$$(d_\theta f)^\mu = \theta^{\mu\nu}\partial_\nu f\,,\quad (d_\theta\xi)^{\mu\nu} = \theta^{\mu\rho}\partial_\rho\xi^\nu - \theta^{\nu\rho}\partial_\rho\xi^\mu - \xi^\rho\partial_\rho\theta^{\mu\nu}\,. \tag{5.20}$$

The generalized formula for the exterior derivative can also be written in terms of the Schouten-Nijenhuis bracket [21]

$$d_\theta = [\theta,.]_S\,. \tag{5.21}$$

The exterior derivative is nilpotent for the Poisson bi-vector θ. We can prove that easily for functions

$$\begin{aligned} (d_\theta d_\theta f)^{\mu\nu} &= \theta^{\mu\rho}\partial_\rho(\theta^{\nu\sigma}\partial_\sigma f) - \theta^{\nu\rho}\partial_\rho(\theta^{\mu\sigma}\partial_\sigma f) - \theta^{\rho\sigma}\partial_\sigma f\partial_\rho\theta^{\mu\nu} \\ &= \partial_\sigma f(\theta^{\mu\rho}\partial_\rho\theta^{\nu\sigma} + \theta^{\nu\rho}\partial_\rho\theta^{\sigma\mu} + \theta^{\sigma\rho}\partial_\rho\theta^{\mu\nu})\,, \end{aligned} \tag{5.22}$$

which goes to zero if and only if the condition (5.10) is satisfied. For a general multi-vector β of rank r, the nilpotence of the exterior derivative d_θ is a consequence of the graded Jacobi identity (4.29)

$$(-1)^{r-1}[\theta,[\theta,\beta]_S]_S - [\theta,[\beta,\theta]_S]_S + (-1)^{r-1}[\beta,[\theta,\theta]_S]_S = 0\,. \tag{5.23}$$

The third term is zero due to condition (5.10). After applying the graded-commutative relation (4.28) to the second term, with the help of definition (5.21), we obtain

$$d_\theta d_\theta \beta = 0\,, \tag{5.24}$$

for any multi-vector β. If the bi-vector θ is not Poisson, and the relation (5.10) does not hold, one can still define the exterior derivative d_θ. The relation (5.23) has the form

$$2 d_\theta d_\theta \beta + [\beta, [\theta, \theta]_S]_S = 0\,, \tag{5.25}$$

so the exterior derivative is no longer nilpotent, but it does satisfy the Leibniz rule.

5.3 Lie bialgebroid

We saw that Lie algebroid can be defined on a tangent bundle, for example, with the Lie bracket as its bracket, and on the cotangent bundle, for example with the Koszul bracket as its bracket. In general, Lie algebroids can be defined on mutually dual bundles. Of particular interest is the case when the exterior derivative corresponding to one algebroid commute with the bracket of the other algebroid, in which case we obtain the Lie bialgebroid [54, 55], for which we provide a definition below:

Definition 12 (Lie bialgebroid)
Let $\Big(V, [,]_L, \rho\Big)$ be a Lie algebroid and supposee that $\Big(V^\star, [,]_{L^\star}, \rho^\star\Big)$ is also a Lie algebroid, where bundles V and $V^\star$ are dual to each other. The structure $(V, V^\star)$ is said to define a Lie bialgebroid if

$$d^\star [\xi_1, \xi_2]_L = [d^\star \xi_1, \xi_2]_S + [\xi_1, d^\star \xi_2]_S\,, \tag{5.26}$$

where $d^\star$ is a Lie algebroid differential of $V^\star$, and ξ_1, ξ_2 are from smooth section of V, and $[,]_S$ is a Schouten-Nijenhuis bracket on a smooth section of multi-vectors $T \bigwedge^p \mathcal{M}$ defined graded symmetric via Leibniz rule.

A simple corollary can be proven - if $(V, V^\star)$ is a Lie bialgebroid, then $(V^\star, V)$ is also a Lie bialgebroid. The condition (5.26) suggests that the exterior derivative related to the one algebroid bracket acts as the graded bracket on the dual bundle. To illustrate this, let us recall two structures - $\Big(T\mathcal{M}, [,]_L, \mathrm{Id}\Big)$, and $\Big(T^\star\mathcal{M}, [,]_\theta, \theta\Big)$ for Poisson bi-vector θ, that we demonstrated are both examples of Lie algebroids. They are defined on mutually dual bundles and in fact, constitute a Lie bialgebroid. The condition (5.26) becomes

$$\begin{aligned}
&d_\theta [\xi_1, \xi_2]_L = [d_\theta \xi_1, \xi_2]_S + [\xi_1, d_\theta \xi_2]_S \\
&[\theta, [\xi_1, \xi_2]_S]_S = [[\theta, \xi_1]_S, \xi_2]_S + [\xi_1, [\theta, \xi_2]_S]_S \\
&[\theta, [\xi_1, \xi_2]_S]_S + [\xi_1, [\xi_2, \theta]_S]_S + [\xi_2, [\theta, \xi_1]_S]_S = 0\,,
\end{aligned} \tag{5.27}$$

where we firstly used (5.20), and then the graded identity (4.28). The final expression is just the graded Jacobi identity (4.29), so we proved the condition (5.26). The exterior derivative related to the Koszul bracket d_θ acts on multi-vectors as the Schouten-Nijenhuis bracket.

The attempts to construct structures similar to Lie algebroids on the double of Lie bialgebroid $V \oplus V^\star$ led to the development of generalized geometry and construction of Courant algebroids, which we will introduce in the next chapter.

Chapter 6

Generalized tangent bundle

In this chapter, we will consider the so-called generalized geometry, that is to say, the geometry of the generalized tangent bundle. Primarily, we will define two natural inner products on the space of generalized vectors. Secondarily, we will obtain the Courant bracket as the extension of the Lie bracket to the section of the generalized tangent bundle. Lastly, we will define the Courant algebroid together with the conditions for it to be the Lie algebroid.

6.1 Inner product and $O(D, D)$ group

The research of generalized geometry was pioneered in the early 2000s, in the works of Hitchin and his introduction of generalized Calabi-Yau manifolds [56], and later in the works of his student Gualtieri [57]. It is a geometry of the generalized tangent bundle, defined as a direct sum of tangent and cotangent bundle over a manifold $T\mathcal{M} \oplus T^\star\mathcal{M}$. The elements of its section are generalized vectors, that have both the vector and 1-form components

$$\Lambda^M = \xi^\mu \oplus \lambda_\mu = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix} , \tag{6.1}$$

where ξ represents the vector components, and λ represents the 1-form components of generalized vectors.

The interior product (4.14) defines a natural way to combine vectors and 1-forms into a scalar $i_\xi \lambda = \xi^\mu \lambda_\mu$. We can use this to define an inner product on the smooth section of the generalized tangent bundle. We can define the inner product between generalized vectors in two ways - so that it is symmetric and antisymmetric. In the former case, it is defined by

$$\langle \Lambda_1, \Lambda_2 \rangle = \langle \xi_1 \oplus \lambda_1, \xi_2 \oplus \lambda_2 \rangle = i_{\xi_1} \lambda_2 + i_{\xi_2} \lambda_1 \, . \tag{6.2}$$

The signature of the symmetric inner product is (D, D), where D is the dimension of the manifold $\mathcal{M}$. It is well known that the Lie group of linear transformations that leaves the inner product of such a signature invariant is the indefinite orthogonal $O(D, D)$ group. This group plays a very important role in string theory, as it governs the T-duality transformations. The fact that a transformation $\mathcal{O}$ keeps the inner product (6.2) invariant can be expressed in matrix notation by

$$(\mathcal{O}^T)_M^{\ P}\, \eta_{PQ}\, \mathcal{O}^Q_{\ N} = \eta_{MN}\,, \tag{6.3}$$

where η is $O(D, D)$ invariant metric, given by

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{6.4}$$

The metric can be used for lowering or raising indices $M,\ N$.

Some of more notable examples of $O(D, D)$ transformations include B-transformations (or B-shifts), which are given by

$$e^{\hat{B}} = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix} \qquad \hat{B}^M_{\ N} = \begin{pmatrix} 0 & 0 \\ 2B_{\mu\nu} & 0 \end{pmatrix} . \tag{6.5}$$

Their inverse is easily obtained from (B.4)

$$e^{-\hat{B}} = \begin{pmatrix} 1 & 0 \\ -2B & 1 \end{pmatrix} . \tag{6.6}$$

The B-transformations act on the generalized metric (2.38) by shifting the Kalb-Ramond field. The second example we will outline here is the θ-transformations, which are given by

$$e^{\hat{\theta}} = \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix} , \quad \hat{\theta}^M_{\ N} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 0 & 0 \end{pmatrix} , \tag{6.7}$$

and when inverted by

$$e^{-\hat{\theta}} = \begin{pmatrix} 1 & -\kappa\theta \\ 0 & 1 \end{pmatrix} . \tag{6.8}$$

It is easy to show that indeed

$$\langle e^{\hat{B}}\Lambda_1, e^{\hat{B}}\Lambda_2\rangle = \langle \Lambda_1, \Lambda_2\rangle\,, \quad \langle e^{\hat{\theta}}\Lambda_1, e^{\hat{\theta}}\Lambda_2\rangle = \langle \Lambda_1, \Lambda_2\rangle\,. \tag{6.9}$$

We will encounter these and other $O(D, D)$ transformations throughout the thesis. For more mathematically rigorous details of $O(D, D)$ group, see Appendix [B].

For completeness, let us also define the antisymmetric inner product by

$$\langle\Lambda_1,\Lambda_2\rangle_- = \langle\xi_1\oplus\lambda_1,\xi_2\oplus\lambda_2\rangle_- = i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1\,. \tag{6.10}$$

As we will see in the next section, this inner product is used in a definition of the Courant bracket. We will primarily be interested in the symmetric inner product, which we will simply refer to as the inner product from now onward.

6.2 Courant bracket

We would like to generalize the notion of the Lie bracket, defined on generalized vectors. The definition of the generalized tangent bundle as a direct sum of two dual bundles $T\mathcal{M}$ and $T^\star\mathcal{M}$ is suitable for considering a Lie bialgebroid structure of the form $(T\mathcal{M},T^\star\mathcal{M})$, with their respective brackets $[,]_L$ and $[,]_{L^\star}$. We can define a skew-symmetric bracket on the smooth section of the generalized tangent bundle by

$$\begin{aligned}[\Lambda_1,\Lambda_2] = \quad &\Big([\xi_1,\xi_2]_L + {\mathcal{L}^\star}_{\lambda_1}\xi_2 - {\mathcal{L}^\star}_{\lambda_2}\xi_1 - \frac{1}{2}d^\star\langle\Lambda_1,\Lambda_2\rangle_-\Big)\\ &\oplus\Big([\lambda_1,\lambda_2]_{L^\star} + \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d\langle\Lambda_1,\Lambda_2\rangle_-\Big)\,.\end{aligned} \tag{6.11}$$

Here, $\mathcal{L}_\xi$ and d represent the Lie derivative and exterior derivative defined on the Lie algebroid on $T\mathcal{M}$, while ${\mathcal{L}^\star}_\lambda$ and $d^\star$ are analogous operations corresponding to the Lie algebroid of $T^\star\mathcal{M}$.

Let us construct a simple example of a skew-symmetric bracket using (6.11). For a bracket on the tangent bundle, we will use the usual Lie bracket. On the cotangent bundle, we can use the trivial bracket that is zero between any two forms. This corresponds to the Lie algebroid with the anchor $\rho^\star = 0$, and the Lie bialgebroid compatibility condition (5.26) is satisfied. The above relation for the bracket gives rise to the well-known Courant bracket given by

$$\begin{aligned}[\Lambda_1,\Lambda_2]_\mathcal{C} &= \xi\oplus\lambda\\ \xi &= [\xi_1,\xi_2]_L\,,\\ \lambda &= \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\,.\end{aligned} \tag{6.12}$$

The Courant bracket is the generalization of the Lie bracket to the generalized tangent bundle. The right-hand side vector and 1-form components of (6.12) can be expressed in some local coordinate basis by

$$\begin{aligned}\xi^\mu &= \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu\,,\\ \lambda_\mu &= \xi_1^\nu(\partial_\nu\lambda_{2\mu} - \partial_\mu\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\lambda_{1\mu} - \partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\lambda_2 - \xi_2\lambda_1)\,.\end{aligned} \tag{6.13}$$

There is a natural way to introduce the projections to the tangent and cotangent bundle π and $\tilde{\pi}$ respectively by

$$\pi(\Lambda) = \pi(\xi \oplus \lambda) = \xi\,, \quad \tilde{\pi}(\Lambda) = \tilde{\pi}(\xi \oplus \lambda) = \lambda\,. \tag{6.14}$$

From the expression (6.12), we see that the Courant bracket on vectors reduces to Lie bracket, while on 1-form it becomes zero

$$[\pi(\Lambda_1), \pi(\Lambda_2)]_{\mathcal{C}} = [\xi_1, \xi_2]_L\,, \quad [\tilde{\pi}(\Lambda_1), \tilde{\pi}(\Lambda_2)]_{\mathcal{C}} = 0\,. \tag{6.15}$$

Effectively, the Courant bracket on smooth sections of tangent and cotangent bundles reduces to the respective Lie algebroid brackets from which it was constructed. Therefore, though the generalized tangent bundle treats vectors and 1-forms in a symmetrical manner, the Courant bracket defined on it does not. Moreover, we have

$$\pi\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}}\Big) = [\pi(\Lambda_1), \pi(\Lambda_2)]_{\mathcal{C}}\,, \quad \tilde{\pi}\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}}\Big) \neq [\tilde{\pi}(\Lambda_1), \tilde{\pi}(\Lambda_2)]_{\mathcal{C}}\,, \tag{6.16}$$

and hence the projection on the tangent bundle is involutive with respect to the Courant bracket, while the projection on the cotangent bundle is not.

In general, the Courant bracket satisfies neither the Leibniz rule nor Jacobi identity (see Appendix [C] for proof). In fact, the deviation from these identities can be expressed in terms of the exterior derivative of the inner product

$$[\Lambda_1, f\Lambda_2]_{\mathcal{C}} = f[\Lambda_1, \Lambda_2]_{\mathcal{C}} + (\mathcal{L}_{\pi(\Lambda_1)} f)\Lambda_2 - \frac{1}{2}\langle \Lambda_1, \Lambda_2 \rangle df\,, \tag{6.17}$$

$$\text{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) = d\text{Nij}(\Lambda_1, \Lambda_2, \Lambda_3)\,, \tag{6.18}$$

where Jac is the Jacobiator, given by

$$\text{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) = [[\Lambda_1, \Lambda_2]\,, \Lambda_3] + [[\Lambda_2, \Lambda_3]\,, \Lambda_1] + [[\Lambda_3, \Lambda_1]\,, \Lambda_2]\,, \tag{6.19}$$

and Nij is the Nijenhuis operator, given by

$$\text{Nij}(\Lambda_1, \Lambda_2, \Lambda_3) = \frac{1}{6}\Big(\langle [\Lambda_1, \Lambda_2], \Lambda_3 \rangle + \langle [\Lambda_2, \Lambda_3], \Lambda_1 \rangle + \langle [\Lambda_3, \Lambda_1], \Lambda_2 \rangle\Big)\,. \tag{6.20}$$

6.3 Courant algebroid

We saw that the Courant bracket does not satisfy the Lie algebroid conditions, and as such, we cannot in general define the Lie algebroid with the Courant bracket as its bracket. However, one can define the Lie algebroid generalization, called Courant algebroid [58]. We provide its definition below:

Definition 13 (Courant algebroid)

Let V be a vector bundle, $\langle , \rangle$ the non-degenerate inner product and $[,]$ a skew-symmetric bracket on a smooth section of a vector bundle V, and let $\rho : V \to TM$ be a smooth bundle map called anchor. Let $\mathcal{D}$ be a differential operator on smooth functions defined by

$$\langle \mathcal{D}f, \Lambda \rangle = \mathcal{L}_{\rho(\Lambda)} f \, . \tag{6.21}$$

The structure $\Big(V, \langle , \rangle, [,], \rho \Big)$ is called the **Courant algebroid** *if it satisfies the following compatibility relations*

$$\rho[\Lambda_1, \Lambda_2] = [\rho(\Lambda_1), \rho(\Lambda_2)]_L \, , \tag{6.22}$$

$$[\Lambda_1, f\Lambda_2] = f[\Lambda_1, \Lambda_2] + (\mathcal{L}_{\rho(\Lambda_1)} f)\Lambda_2 - \frac{1}{2}\langle \Lambda_1, \Lambda_2 \rangle \mathcal{D}f \, , \tag{6.23}$$

$$\mathcal{L}_{\rho(\Lambda_1)} \langle \Lambda_2, \Lambda_3 \rangle = \langle [\Lambda_1, \Lambda_2] + \frac{1}{2}\mathcal{D}\langle \Lambda_1, \Lambda_2 \rangle, \Lambda_3 \rangle + \langle \Lambda_2, [\Lambda_1, \Lambda_3] + \frac{1}{2}\mathcal{D}\langle \Lambda_1, \Lambda_3 \rangle \rangle \, , \tag{6.24}$$

$$\langle \mathcal{D}f, \mathcal{D}g \rangle = 0 \, , \tag{6.25}$$

$$\mathit{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) = \mathcal{D}\mathit{Nij}(\Lambda_1, \Lambda_2, \Lambda_3) \, , \tag{6.26}$$

for all Λ_1, Λ_2, Λ_3 from smooth section of a vector bundle V, and for all smooth functions f and g on the manifold.

Any double of Lie bialgebroid defines the Courant algebroid. The reverse however is not the case, and the Courant algebroid brackets encompass a larger set of brackets [59].

A straightforward example of Courant algebroids, that we will refer to as the standard Courant algebroid, consists of the generalized tangent bundle, the $O(D, D)$ invariant inner product (6.2), the projection π from the generalized tangent bundle to the tangent bundle as its anchor (6.14), and the Courant bracket (6.12). It is not difficult to show that the five compatibility conditions are satisfied. We present the proof in the Appendix [C].

There is an alternative definition of the Courant algebroid, in which the Courant algebroid bracket is defined so that it is not skew-symmetric, but it does satisfy the Leibniz rule and Jacobi identity [60]. This definition is proven to be equivalent to the one provided above. The skew-symmetric brackets are more suitable to describe the algebra of symmetries, and we will use the definition (Def. 13) exclusively in this thesis.

Perhaps the most striking application of Courant algebroids is in the attempt to explain the T-duality. For instance, in [61, 62], authors investigated two manifolds $\mathcal{M}$ and $\tilde{\mathcal{M}}$, both principle torus bundles over a common manifold B. The manifolds had 3-form fluxes H and $\tilde{H}$ on them. The conditions for the theories defined on these two manifolds with fluxes to be mutually T-dual were obtained, which were shown in [63] to be equivalent to the isomorphism ϕ between two Courant algebroids, that satisfies

$$\langle\phi(\Lambda_1),\phi(\Lambda_2)\rangle=\langle\Lambda_1,\Lambda_2\rangle\,,\quad \phi\Big([\Lambda_1,\Lambda_2]_{\mathcal{C}_H}\Big)=[\phi(\Lambda_1),\phi(\Lambda_2)]_{\mathcal{C}_{\tilde{H}}}\,, \tag{6.27}$$

where $[,]_{\mathcal{C}_H}$ denotes the Courant bracket deformed with the flux H. As such, the Courant algebroids appear convenient to describe T-duality. We will further investigate it in the analysis of bosonic string σ-model symmetries.

6.4 Dirac structures

Definition 14 (Dirac structures)

For a sub-bundle to be isotropic with respect to the inner product means that the inner product of any two generalized vectors from its section is zero

$$\langle\Lambda_1,\Lambda_2\rangle=0\,. \tag{6.28}$$

Dirac structures are defined as the isotropic sub-bundles with the maximal dimension that are closed under the skew-symmetric Courant algebroid bracket.

For any 2-form B, the sub-bundle

$$\mathcal{V}_B(\Lambda)=\xi^\mu\oplus 2B_{\mu\nu}\xi^\nu \tag{6.29}$$

is going to be isotropic with respect to the $O(D,D)$ invariant inner product (6.2) due to the antisymmetric properties of a 2-form

$$\langle\xi_1^\mu\oplus 2B_{\mu\rho}\xi_1^\rho,\xi_2^\nu\oplus 2B_{\nu\sigma}\xi_2^\sigma\rangle=2B_{\mu\nu}(\xi_1^\mu\xi_2^\nu+\xi_1^\nu\xi_2^\mu)=0\,. \tag{6.30}$$

Similarly, for any bi-vector θ, we can construct another isotropic sub-bundle by

$$\mathcal{V}_\theta(\Lambda)=\kappa\theta^{\mu\nu}\lambda_\nu\oplus\lambda_\mu\,. \tag{6.31}$$

The sub-bundle of the form $\mathcal{V}_B$ and $\mathcal{V}_\theta$ from the mathematical perspective represent a graph of 2-form over tangent bundle, and a graph of bi-vector over a cotangent bundle, respectively.

The importance of Dirac structures lies in the fact that the Courant algebroid on them reduces to the Lie algebroid. To see this, let us take a look at the second Courant algebroid compatibility condition (6.23), and note that the algebroid bracket satisfies Leibniz rule up to the term $\frac{1}{2}\langle\Lambda_1,\Lambda_2\rangle\mathcal{D}f$. This term is zero on isotropic sub-spaces (6.28), and therefore the Leibniz identity will be satisfied on them. Likewise, the fifth compatibility condition (6.26) becomes the usual Jacobi identity on Dirac structures. This can be deduced by evaluating the Nijenhuis operator (6.20) of some vectors from the Dirac structure $\mathcal{V}$. Without loss of generality, we have

$$\begin{aligned} &\Lambda_1, \Lambda_2 \in \mathcal{V} \Rightarrow [\Lambda_1, \Lambda_2] \in \mathcal{V}\,, \\ &[\Lambda_1, \Lambda_2], \Lambda_3 \in \mathcal{V} \Rightarrow \langle [\Lambda_1, \Lambda_2], \Lambda_3 \rangle = 0\,, \end{aligned} \tag{6.32}$$

where the first line is the consequence of Dirac structures being closed under the bracket, and the second line stands from the definition of isotropic spaces (6.28).

6.4.1 Dirac structures of the standard Courant algebroid

Let us calculate the Dirac structures associated with the standard Courant bracket. Substituting $\lambda_{1\mu} = 2B_{\mu\nu}\xi_1^\nu$ and $\lambda_{2\mu} = 2B_{\mu\nu}\xi_2^\nu$ into the second relation of (6.13), we obtain

$$\begin{aligned} \lambda_\mu &= 2\xi_1^\nu\Big(\partial_\nu(B_{\mu\rho}\xi_2^\rho) - \partial_\mu(B_{\nu\rho}\xi_2^\rho)\Big) - 2\xi_2^\nu\Big(\partial_\nu(B_{\mu\rho}\xi_1^\rho) - \partial_\mu(B_{\nu\rho}\xi_1^\rho)\Big) + 2\partial_\mu(B_{\nu\rho}\xi_1^\nu\xi_2^\rho) \\ &= 2B_{\mu\rho}(\xi_1^\nu\partial_\nu\xi_2^\rho - \xi_2^\nu\partial_\nu\xi_1^\rho) + 2B_{\nu\rho}(\xi_2^\nu\partial_\mu\xi_1^\rho + \xi_2^\rho\partial_\mu\xi_1^\nu) \\ &\quad +2\xi_1^\nu\xi_2^\rho(\partial_\nu B_{\mu\rho} - \partial_\mu B_{\nu\rho} - \partial_\rho B_{\mu\nu} + \partial_\mu B_{\rho\nu} + \partial_\mu B_{\nu\rho}) \\ &= 2B_{\mu\nu}\xi^\nu - 2(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})\xi_1^\nu\xi_2^\rho\,, \end{aligned} \tag{6.33}$$

where we first applied the chain rule, and then the skew-symmetric properties of B, together with expressing the first relation of (6.13) to express ξ^μ. Therefore, the sub-bundle $\mathcal{V}_B$ (6.29) is a Dirac structure for a closed 2-form B

$$dB = 0 \Leftrightarrow \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} = 0\,. \tag{6.34}$$

Mathematically, the manifold $\mathcal{M}$ with a closed non-degenerate 2-form B is a symplectic structure. If we interpret a 2-form B as the Kalb-Ramond field, we obtained that its field strength is zero, or equivalently that H-flux is zero.

On the other hand, for sub-bundle $\mathcal{V}_\theta$ we firstly note that it can be written as

$$\mathcal{V}_\theta(\Lambda) = -\theta(\lambda) \oplus \lambda\,, \tag{6.35}$$

where $\theta(\lambda)$ is defined as in (5.9). For $\mathcal{V}_\theta$ to be closed under the Courant bracket, we can use the coordinate-free expression for the Courant bracket (6.12), and obtain

$$\begin{aligned} \xi &= [\theta(\lambda_1), \theta(\lambda_2)]_L \\ \lambda &= -\Big(\mathcal{L}_{\theta(\lambda_1)}\lambda_2 - \mathcal{L}_{\theta(\lambda_2)}\lambda_1 - d(\theta(\lambda_1, \lambda_2))\Big) = -[\lambda_1, \lambda_2]_\theta\,, \end{aligned} \tag{6.36}$$

where we recognized in the second line the expression for the Koszul bracket (5.11). We can use the relation (5.15) which is correct only for $[\theta, \theta]_S = 0$ (5.10). On this Dirac structure, we obtained the Poisson manifold [A]. If the bi-vector θ is interpreted as the non-commutative parameter (3.16), this translates into the R-flux being zero.

In general, Dirac structures define integrability conditions for Courant algebroids. We saw how symplectic and Poisson manifolds are obtained from the Dirac structures for the standard Courant algebroid. Moreover, we observed that Dirac structures associated with the standard Courant algebroid put severe restrictions on string fluxes. In the following chapters of the thesis, we will investigate twisted Courant brackets and their corresponding Courant algebroids. We will demonstrate that these restrictions on string fluxes are relaxed on Dirac structures related to the twisted Courant algebroids [64].

Part III

Single theory

Chapter 7

Symmetries of bosonic string

In this chapter, we will obtain generators of both diffeomorphisms and local gauge transformations for the bosonic string σ-model. We will show that these symmetries are not independent, rather they are related by T-duality. In the end, we will consider the double generator governing both of these symmetries and show that its Poisson bracket algebra produces the Courant bracket.

7.1 Symmetry generators

Symmetry is generally understood as a change in space-time fields that does not change the physically observable quantities. In the standard approach of quantum field theory, symmetries can be seen as transformations of the background fields that keep classical action invariant. In string theory, background fields are defined as functions on the world-sheet, which possess conformal invariance. The physical observable quantities, like scattering amplitudes, are obtained from conformal field theory. Therefore, symmetries imply the existence of physically equivalent solutions to the string equations of motions, which correspond to the mutually isomorphic conformal field theories [65, 66].

Symmetries in string theory σ-model are governed by generators. They are scalars $\mathcal{G}$, that in classical theory act on Hamiltonian via Poisson bracket. If the Poisson bracket between the generator and Hamiltonian can be interpreted as the change in fields, we say that $\mathcal{G}$ generates a symmetry, i.e. if

$$\mathcal{H}_{(G,B)} + \{\mathcal{G}, \mathcal{H}_{(G,B)}\} = \mathcal{H}_{(G+\delta G, B+\delta B)} \,, \tag{7.1}$$

we say that $B \to B + \delta B$, and $G \to G + \delta G$ are symmetry transformations of the background fields generated by $\mathcal{G}$. We will seek symmetry generators $\mathcal{G}$ such that its action on Hamiltonian seeks a change that can be interpreted as the difference in background fields.

7.1.1 Diffeomorphisms

The first generator to be considered will be in the form [67]

$$\mathcal{G}_\xi = \int_0^{2\pi} d\sigma \xi^\mu(x(\sigma))\pi_\mu(\sigma)\,, \tag{7.2}$$

with ξ^μ being a symmetry parameter. The usual equal-time Poisson bracket relations are assumed

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta(\sigma-\bar\sigma)\,, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = \{x^\mu(\sigma), x^\nu(\bar\sigma)\} = 0 \ . \tag{7.3}$$

The change in Hamiltonian generated by $\mathcal{G}_\xi$ (7.2) can be expressed as

$$\{\mathcal{G}_\xi, \mathcal{H}\} = \delta_\xi \mathcal{H}\,. \tag{7.4}$$

The transformation of each term in the Hamiltonian (2.35) will be considered separately. For the change in the first term, we have

$$\begin{aligned}
\delta_\xi\Big(\frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu\Big) &= \frac{1}{2\kappa}\int d\bar\sigma \Big\{\xi^\rho(\bar\sigma)\pi_\rho(\bar\sigma), \pi_\mu(\sigma)(G^{-1})^{\mu\nu}(\sigma)\pi_\nu(\sigma)\Big\} \\
&= \frac{1}{2\kappa}\int d\bar\sigma \Big(\pi_\rho \partial_\mu \xi^\rho (G^{-1})^{\mu\nu}\pi_\nu + \pi_\mu (G^{-1})^{\mu\nu}\partial_\nu \xi^\rho \pi_\rho \\
&\qquad -\pi_\mu \xi^\rho \partial_\rho (G^{-1})^{\mu\nu}\pi_\nu\Big)\delta(\sigma-\bar\sigma) \\
&= \frac{1}{2\kappa}\pi_\mu\Big(-\xi^\rho\partial_\rho (G^{-1})^{\mu\nu} + \partial_\rho \xi^\mu (G^{-1})^{\nu\rho} + (G^{-1})^{\mu\rho}\partial_\rho\xi^\nu\Big)\pi_\nu\,,
\end{aligned} \tag{7.5}$$

where in the second step we omitted dependence on σ, and relabeled some dummy indices, to make the expression more readable. For the second term of Hamiltonian (2.35), we write

$$\begin{aligned}
\delta_\xi\Big(\frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu\Big) &= \frac{\kappa}{2}\int d\bar\sigma \Big\{\xi^\rho(\bar\sigma)\pi_\rho(\bar\sigma), x'^\mu(\sigma)G^E_{\mu\nu}(\sigma)x'^\nu(\sigma)\Big\} \\
&= \frac{\kappa}{2}\int d\bar\sigma \Big[\Big(\xi^\mu(\bar\sigma)G^E_{\mu\nu}x'^\nu + x'^\mu G^E_{\mu\nu}\xi^\nu(\bar\sigma)\Big)\delta'(\sigma-\bar\sigma) - x'^\mu\xi^\rho\partial_\rho G^E_{\mu\nu}x'^\nu\delta(\sigma-\bar\sigma)\Big] \\
&= \frac{\kappa}{2}\int d\bar\sigma \Big[\Big(x'^\rho\partial_\rho\xi^\mu G^E_{\mu\nu}x'^\nu + x'^\mu G^E_{\mu\nu}x'^\rho\partial_\rho\xi^\nu - x'^\mu\xi^\rho\partial_\rho G^E_{\mu\nu}x'^\nu\Big)\delta(\sigma-\bar\sigma) \\
&\qquad + \Big(\xi^\mu G^E_{\mu\nu}x'^\nu + x'^\mu G^E_{\mu\nu}\xi^\nu\Big)\delta'(\sigma-\bar\sigma)\Big] \\
&= \frac{\kappa}{2}x'^\mu\Big(\partial_\mu\xi^\rho G^E_{\rho\nu} + G^E_{\mu\rho}\partial_\nu\xi^\rho - \xi^\rho\partial_\rho G^E_{\mu\nu}\Big)x'^\nu\,,
\end{aligned} \tag{7.6}$$

where in the third line we used the property of the delta function

$$f(\bar\sigma)\delta'(\sigma-\bar\sigma) = f'(\sigma)\delta(\sigma-\bar\sigma) + f(\sigma)\delta'(\sigma-\bar\sigma)\,, \tag{7.7}$$

and in the last line, we used

$$\int d\bar{\sigma} f(\sigma)\delta'(\sigma-\bar{\sigma}) = f(\sigma)\frac{\partial}{\partial_\sigma}\int d\bar{\sigma}\delta(\sigma-\bar{\sigma}) = 0\,, \tag{7.8}$$

which makes the anomalous part become zero in (7.6).

The last term in the Hamiltonian (2.35) transforms as

$$\begin{aligned}
\delta_\xi\Big(-2x'^\mu(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\Big) &= -2\int d\bar{\sigma}\Big\{\xi^\rho(\bar{\sigma})\pi_\rho(\bar{\sigma}), x'^\mu(\sigma)(BG^{-1})_\mu^{\ \nu}(\sigma)\,\pi_\nu(\sigma)\Big\} \\
&= -2\int d\bar{\sigma}\Big[\Big(x'^\mu(BG^{-1})_\mu^{\ \nu}\,\partial_\nu\xi^\rho\pi_\rho - x'^\mu\xi^\rho\partial_\rho(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\Big)\delta(\sigma-\bar{\sigma}) \\
&\qquad +\xi^\mu(\bar{\sigma})(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\delta'(\sigma-\bar{\sigma})\Big] \\
&= -2\int d\bar{\sigma}\Big[\Big(x'^\mu(BG^{-1})_\mu^{\ \nu}\,\partial_\nu\xi^\rho\pi_\rho - x'^\mu\xi^\rho\partial_\rho(BG^{-1})_\mu^{\ \nu}\,\pi_\nu \\
&\qquad +x'^\rho\partial_\rho\xi^\mu(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\Big)\delta(\sigma-\bar{\sigma}) + \xi^\mu(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\delta'(\sigma-\bar{\sigma})\Big] \\
&= -2x'^\mu\Big((BG^{-1})_\mu^{\ \rho}\,\partial_\rho\xi^\nu + \partial_\mu\xi^\rho(BG^{-1})_\rho^{\ \nu} - \xi^\rho\partial_\rho(BG^{-1})_\mu^{\ \nu}\Big)\pi_\nu\,,
\end{aligned} \tag{7.9}$$

where we once again used (7.7) and (7.8). Substituting (7.5), (7.6) and (7.9) into (7.4), we can read the following transformation laws

$$\delta_\xi(G^{-1})^{\mu\nu} = -\xi^\rho\partial_\rho(G^{-1})^{\mu\nu} + \partial_\rho\xi^\mu(G^{-1})^{\nu\rho} + (G^{-1})^{\mu\rho}\partial_\rho\xi^\nu\,, \tag{7.10}$$

$$\delta_\xi G^E_{\mu\nu} = \partial_\mu\xi^\rho G^E_{\rho\nu} + G^E_{\mu\rho}\partial_\nu\xi^\rho - \xi^\rho\partial_\rho G^E_{\mu\nu}\,, \tag{7.11}$$

$$\delta_\xi(BG^{-1})_\mu^{\ \nu} = (BG^{-1})_\mu^{\ \rho}\,\partial_\rho\xi^\nu + \partial_\mu\xi^\rho(BG^{-1})_\rho^{\ \nu} - \xi^\rho\partial_\rho(BG^{-1})_\mu^{\ \nu}\,. \tag{7.12}$$

From these relations, we can easily obtain the transformation laws for metric and Kalb-Ramond tensor. For instance, using

$$\delta_\xi(G_{\mu\rho}(G^{-1})^{\rho\nu}) = \delta_\xi G_{\mu\rho}(G^{-1})^{\rho\nu} + G_{\mu\rho}\delta_\xi(G^{-1})^{\rho\nu} = 0\,, \tag{7.13}$$

we obtain

$$\delta_\xi G_{\mu\nu} = -\xi^\rho\partial_\rho G_{\mu\nu} - \partial_\mu\xi^\rho G_{\rho\nu} - \partial_\nu\xi^\rho G_{\rho\mu}\,. \tag{7.14}$$

Similarly, substituting $\delta_\xi(B_{\mu\rho}(G^{-1})^{\rho\nu}) = \delta_\xi B_{\mu\rho}(G^{-1})^{\rho\nu} + B_{\mu\rho}\delta_\xi(G^{-1})^{\rho\nu}$ into (7.12), we obtain

$$\delta_\xi B_{\mu\nu} = -\xi^\rho\partial_\rho B_{\mu\nu} + \partial_\mu\xi^\rho B_{\rho\nu} - B_{\mu\rho}\partial_\nu\xi^\rho\,. \tag{7.15}$$

Without loss of generality, we can change the sign of the parameter $\xi \to -\xi$, and write

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu} = \xi^\rho\partial_\rho G_{\mu\nu} + \partial_\mu\xi^\rho G_{\rho\nu} + \partial_\nu\xi^\rho G_{\rho\mu}\,, \tag{7.16}$$

$$\delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} = \xi^\rho\partial_\rho B_{\mu\nu} - \partial_\mu\xi^\rho B_{\rho\nu} + B_{\mu\rho}\partial_\nu\xi^\rho\,. \tag{7.17}$$

These are the general coordinate transformations or diffeomorphisms. The Poisson bracket satisfies the Jacobi identity, so we can write

$$\begin{aligned} 0 &= \{\mathcal{G}_{\xi_1},\{\mathcal{G}_{\xi_2},\mathcal{H}\}\}+\{\mathcal{G}_{\xi_2},\{\mathcal{H},\mathcal{G}_{\xi_1},\}\}+\{\mathcal{H},\{\mathcal{G}_{\xi_1},\mathcal{G}_{\xi_2}\}\} \\ &= \{\mathcal{G}_{\xi_1},\{\mathcal{G}_{\xi_2},\mathcal{H}\}\}-\{\mathcal{G}_{\xi_2},\{\mathcal{G}_{\xi_1},\mathcal{H}\}\}-\{\{\mathcal{G}_{\xi_1},\mathcal{G}_{\xi_2}\},\mathcal{H}\}\,, \end{aligned} \tag{7.18}$$

from which we obtain that the algebra of generators governing diffeomorphisms closes on the Lie bracket (4.5)

$$\Big\{\mathcal{G}_{\xi_1}\,,\mathcal{G}_{\xi_2}\Big\}=-\mathcal{G}_{[\xi_1,\xi_2]_L}\,. \tag{7.19}$$

7.1.2 Local gauge transformations

We now seek the generator in the form

$$\mathcal{G}_\lambda=\int_0^{2\pi}d\sigma\lambda_\mu(x(\sigma))\kappa x'^\mu(\sigma)\,, \tag{7.20}$$

so that its action on the Hamiltonian via Poisson bracket

$$\{\mathcal{G}_\lambda,\mathcal{H}\}=\delta_\lambda\mathcal{H}\,. \tag{7.21}$$

can be interpreted as the change of background fields. With the help of delta function identity (7.7), we obtain

$$\begin{aligned} \delta_\lambda\Big(\frac{1}{2\kappa}\pi_\mu(G^{-1})^{\mu\nu}\pi_\nu\Big) &= \frac{1}{2\kappa}\int d\bar\sigma\Big\{\kappa\lambda_\rho(\bar\sigma)\partial_{\bar\sigma}x^\rho(\bar\sigma),\pi_\mu(\sigma)(G^{-1})^{\mu\nu}(\sigma)\pi_\nu(\sigma)\Big\} \\ &= \int d\bar\sigma\Big(-\lambda_\mu(\bar\sigma)(G^{-1})^{\mu\nu}\pi_\nu\delta'(\sigma-\bar\sigma)+x'^\rho\partial_\mu\lambda_\rho(G^{-1})^{\mu\nu}\pi_\nu\delta(\sigma-\bar\sigma)\Big) \\ &= x'^\mu(\partial_\rho\lambda_\mu-\partial_\mu\lambda_\rho)(G^{-1})^{\rho\nu}\pi_\nu\,, \end{aligned} \tag{7.22}$$

where the anomalous part goes to zero due to (7.8). The second Hamiltonian term (2.35) does not depend on momenta, so we have

$$\delta_\lambda\Big(\frac{1}{2}x'^\mu G^E_{\mu\nu}x'^\nu\Big)=\frac{1}{2}\int d\bar\sigma\Big\{\lambda_\rho(\bar\sigma)\partial_{\bar\sigma}x^\rho(\bar\sigma),x'^\mu(\sigma)G^E_{\mu\nu}(\sigma)x'^\nu(\sigma)\Big\}=0\,, \tag{7.23}$$

and similarly using (7.7), we have

$$\begin{aligned} \delta_\lambda\Big(-2x'^\mu(BG^{-1})_\mu{}^\nu\pi_\nu\Big) &= -2\kappa\int d\bar\sigma\Big\{\lambda_\rho(\bar\sigma)\partial_{\bar\sigma}x^\rho(\bar\sigma),x'^\mu(\sigma)(BG^{-1})_\mu{}^\nu(\sigma)\pi_\nu(\sigma)\Big\} \\ &= -2\kappa x'^\mu(BG^{-1})_\mu{}^\nu\int d\bar\sigma\Big(-\lambda_\nu(\bar\sigma)\delta'(\sigma-\bar\sigma)+\partial_\nu\lambda_\rho x'^\rho\delta(\sigma-\bar\sigma)\Big) \\ &= 2\kappa x'^\mu(BG^{-1})_\mu{}^\rho(\partial_\nu\lambda_\rho-\partial_\rho\lambda_\nu)x'^\nu\,. \end{aligned} \tag{7.24}$$

Substituting contributions (7.22), (7.23) and (7.24) into (7.21), we obtain

$$\delta_\lambda (G^{-1})^{\mu\nu} = 0\,, \tag{7.25}$$
$$\delta_\lambda G^E_{\mu\nu} = 4(BG^{-1})_\mu{}^\rho(\partial_\rho\lambda_\nu - \partial_\nu\lambda_\rho)\,, \tag{7.26}$$
$$\delta_\lambda (BG^{-1})_\mu{}^\nu = \frac{1}{2}(\partial_\mu\lambda_\rho - \partial_\rho\lambda_\mu)(G^{-1})^{\rho\nu}\,, \tag{7.27}$$

from which we read the transformation of background fields

$$\begin{aligned} \delta_\lambda G_{\mu\nu} &= 0\,, \\ \delta_\lambda B_{\mu\nu} &= (d\lambda)_{\mu\nu} = \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu\,, \end{aligned} \tag{7.28}$$

where without loss of generality we redefined $\lambda \to \frac{1}{2}\lambda$. These transformations of the background fields are known as local gauge transformations. There are analogous to the gauge transformations of the vector potential in electromagnetism (2.8).

Local gauge transformations are reducible transformations, due to the nilpotency of the exterior derivative. To demonstrate this, we consider the transformation of the Kalb-Ramond field governed by the generator of local gauge transformations with the parameter that is a sum of parameter λ and the exterior derivative of some smooth function f

$$\delta_{\lambda+df} B = d\lambda + d^2 f = \delta_\lambda B\,, \tag{7.29}$$

or in some coordinate basis

$$\delta_{\lambda+df} B_{\mu\nu} = \partial_\mu(\lambda_\nu + \partial_\nu f) - \partial_\nu(\lambda_\mu + \partial_\mu f) = \delta_\lambda B_{\mu\nu}\,. \tag{7.30}$$

The T-duality exchanges momenta with the winding numbers (3.28). Since canonical momenta and the coordinate σ-derivatives are also generators of the diffeomorphisms and local gauge transformations, respectively, we conclude that the general coordinate transformations and local gauge transformations are not independent, rather they are related by T-duality.

7.2 Double generator and Courant bracket

The mutual relation of generators by T-duality motivates us to consider a single generator that will govern both of these symmetry transformations. The parameters of diffeomorphism ξ^μ are vector components, while the parameters of the local gauge transformations λ_μ are components of the 1-forms. Therefore, we can combine two parameters into a generalized vector Λ^M, where

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix}\,. \tag{7.31}$$

The generator governing both symmetry transformations is just the sum of generators $\mathcal{G}_\xi$ (7.2) and $\mathcal{G}_\lambda$ (7.20)

$$\mathcal{G}_\Lambda = \mathcal{G}_\xi + \mathcal{G}_\lambda = \int_0^{2\pi} d\sigma \Big(\xi^\mu \pi_\mu + \lambda_\mu \kappa x'^\mu \Big) \,, \tag{7.32}$$

which can be recognized as the inner product (6.2) on the generalized tangent bundle

$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, X \rangle \,. \tag{7.33}$$

Let us now proceed with the algebra of double generator $\mathcal{G}_\Lambda$. Using the Poisson bracket relations (7.3), we obtain

$$\begin{aligned} \Big\{ \mathcal{G}(\xi_1 \oplus \lambda_1),\, \mathcal{G}(\xi_2 \oplus \lambda_2) \Big\} = & \int d\sigma \Big(\pi_\mu (\xi_2^\nu \partial_\nu \xi_1^\mu - \xi_1^\nu \partial_\nu \xi_2^\mu) + \kappa x'^\mu (\xi_2^\nu \partial_\nu \lambda_{1\mu} - \xi_1^\nu \partial_\nu \lambda_{2\mu}) \Big) \\ & + \int d\sigma d\bar\sigma \kappa \Big(\lambda_{1\mu}(\sigma) \xi_2^\mu(\bar\sigma) + \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\sigma) \Big) \delta'(\sigma - \bar\sigma) \,, \end{aligned} \tag{7.34}$$

where we adopt the notation in which ξ_1 and λ_1 are vector and 1-form components of a generalized vector Λ_1, etc. To transform the anomalous terms, we note the identity

$$\delta'(\sigma - \bar\sigma) = -\partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \,. \tag{7.35}$$

Now, the first term in the last line on the right-hand side of the equation (7.34) can be rewritten as

$$\begin{aligned} \kappa \int d\sigma d\bar\sigma \lambda_{1\mu}(\sigma) \xi_2^\mu(\bar\sigma) \delta'(\sigma - \bar\sigma) &= \frac{\kappa}{2} \int d\sigma d\bar\sigma \Big(\lambda_{1\mu}(\sigma) \xi_2^\mu(\bar\sigma) \delta'(\sigma - \bar\sigma) - \lambda_{1\mu}(\sigma) \xi_2^\mu(\bar\sigma) \partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \Big) \\ &= \frac{\kappa}{2} \int d\sigma d\bar\sigma \Big(\lambda_{1\mu} \xi_2^\mu \delta'(\sigma - \bar\sigma) - \lambda_{1\mu}(\bar\sigma) \xi_2^\mu(\bar\sigma) \partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \Big) \\ &\quad + \frac{\kappa}{2} \int d\sigma \Big(\lambda_{1\mu} \partial_\nu \xi_2^\mu x'^\nu - \partial_\nu \lambda_{1\mu} \xi_2^\mu x'^\nu \Big) \\ &= \kappa \int d\sigma \Big(\frac{1}{2} \partial_\nu (\lambda_{1\mu} \xi_2^\mu) - \xi_2^\mu \partial_\nu \lambda_{1\mu} \Big) x'^\nu \,. \end{aligned} \tag{7.36}$$

We first made the relation symmetric with the help of (7.35). Afterward, we used the relation (7.7), and in the end, we used the chain rule, and (7.8).

Similarly, the second anomalous term in (7.34) is as easily transformed

$$\begin{aligned} \kappa \int d\sigma d\bar\sigma \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\sigma) \delta'(\sigma - \bar\sigma) &= \frac{\kappa}{2} \int d\sigma d\bar\sigma \Big(\lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\sigma) \delta'(\sigma - \bar\sigma) - \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\sigma) \partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \Big) \\ &= \frac{\kappa}{2} \int d\sigma d\bar\sigma \Big(\lambda_{2\mu} \xi_1^\mu \delta'(\sigma - \bar\sigma) - \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\bar\sigma) \partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \Big) \\ &\quad + \frac{\kappa}{2} \int d\sigma \Big(\partial_\nu \lambda_{2\mu} \xi_1^\mu x'^\nu - \lambda_{2\mu} \partial_\nu \xi_1^\mu x'^\nu \Big) \\ &= \kappa \int d\sigma \Big(\xi_1^\mu \partial_\nu \lambda_{2\mu} - \frac{1}{2} \partial_\nu (\lambda_{2\mu} \xi_1^\mu) \Big) x'^\nu \,. \end{aligned} \tag{7.37}$$

Substituting (7.36) and (7.37) into (7.34), we obtain

$$\Big\{\mathcal{G}_{\Lambda_1},\,\mathcal{G}_{\Lambda_2}\Big\} = -\mathcal{G}_{\Lambda}\,, \tag{7.38}$$

where the resulting gauge parameters are given by

$$\begin{aligned} \xi^\mu &= \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu\,, \\ \lambda_\mu &= \xi_1^\nu(\partial_\nu \lambda_{2\mu} - \partial_\mu \lambda_{2\nu}) - \xi_2^\nu(\partial_\nu \lambda_{1\mu} - \partial_\mu \lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\lambda_2 - \xi_2\lambda_1)\,. \end{aligned} \tag{7.39}$$

These relations can be recognized from the previous chapter, as relations (6.13) defining the Courant bracket

$$\Big\{\mathcal{G}_{\Lambda_1},\,\mathcal{G}_{\Lambda_2}\Big\} = -\mathcal{G}_{[\Lambda_1,\Lambda_2]_\mathcal{C}}\,. \tag{7.40}$$

We see that by extending the generator of diffeomorphisms with the local gauge transformations, the Courant bracket appears [68]. Due to the T-dual relation between two symmetry transformations, we conclude that the Courant bracket is the T-dual extension of the Lie bracket.

The Courant bracket, together with the generalized tangent bundle and its symmetric bilinear form (6.2), and the natural projection to the tangent bundle as an anchor, defines the standard Courant algebroid (see Appendix [C]). In the previous chapter, we obtained its Dirac structures and saw that it puts a severe restriction on H- and R-fluxes. In the next chapter, we will introduce the twisted Courant algebroid, that is to say, the Courant algebroid defined with the twisted version of the Courant bracket.

Chapter 8

Twisted Courant algebroid

We will define the Courant bracket twisted by any element of the $O(D,D)$ group and then show that it defines a Courant algebroid, where all the compatibility conditions are a priori satisfied.

8.1 Twisted Courant bracket

Let e^T be an $O(D,D)$ transformation, keeping the inner product (6.2) invariant. Its action on the basis X^M (2.39) produces another basis

$$\hat{X}^M = (e^T)^M_{\ N}\, X^N\,. \tag{8.1}$$

We can express the generator $\mathcal{G}_\Lambda$ (7.33) in this basis, using the invariance of the inner product with respect to e^T

$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, X\rangle = \int d\sigma \langle e^T\Lambda, e^T X\rangle = \int d\sigma \langle \hat{\Lambda}, \hat{X}\rangle = \mathcal{G}^{(T)}_{\hat{\Lambda}}\,, \tag{8.2}$$

where we marked the resulting generator as $\mathcal{G}^{(T)}_{\hat{\Lambda}}$, and where

$$\hat{\Lambda}^M = (e^T)^M_{\ N}\, \Lambda^N\,. \tag{8.3}$$

Using the $O(D,D)$ invariance of the inner product, the algebra relations of the generator written in a new basis becomes

$$\begin{aligned}\left\{\mathcal{G}^{(T)}_{\hat{\Lambda}_1}, \mathcal{G}^{(T)}_{\hat{\Lambda}_2}\right\} &= -\int d\sigma \langle [\Lambda_1,\Lambda_2]_C, X\rangle = -\int d\sigma \langle [e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_C, e^{-T}\hat{X}\rangle \\ &= -\int d\sigma \langle e^T[e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}}, \hat{X}\rangle = -\mathcal{G}^{(T)}_{[\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathcal{C}_T}}\,,\end{aligned} \tag{8.4}$$

where we defined the T-twisted Courant bracket by

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T} = e^T[e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}}\,. \tag{8.5}$$

For each $O(D,D)$ invariant transformation e^T, there is a corresponding twisted Courant bracket. A straightforward method of obtaining the twisted Courant bracket involves using the transformation e^T to change the basis in which the generator is represented, followed by calculating the Poisson bracket algebra of said generator.

8.2 Courant algebroid related to the twisted Courant bracket

Let us demonstrate that the twisted Courant bracket defines a Courant algebroid. We are looking for an anchor that satisfies

$$\rho\Big([\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_\mathcal{T}}\Big) = [\rho(\hat{\Lambda}_1), \rho(\hat{\Lambda}_2)]_L \,. \tag{8.6}$$

Using (8.3) and (8.5), we rewrite the previous relation as

$$\rho\Big(e^T[\Lambda_1, \Lambda_2]_\mathcal{C}\Big) = [\rho(e^T\Lambda_1), \rho(e^T\Lambda_2)]_L \,. \tag{8.7}$$

Now from the fact that the natural projection π (6.14) is the anchor for the standard Courant algebroid (6.16), we obtain

$$\rho(\hat{\Lambda}) = \pi(e^{-T}\hat{\Lambda}) \,. \tag{8.8}$$

The corresponding differential operator is obtained from substituting (8.3) and (8.8) into the definition of the Courant algebroid differential operator (6.21)

$$\langle \mathcal{D}f, \hat{\Lambda}\rangle = \mathcal{L}_{\rho(\hat{\Lambda})}\, f = \mathcal{L}_{\pi(\Lambda)} f = \langle \mathcal{D}^{(0)} f, \Lambda\rangle = \langle \mathcal{D}^{(0)} f, e^{-T}\hat{\Lambda}\rangle = \langle e^T \mathcal{D}^{(0)} f, \hat{\Lambda}\rangle \,, \tag{8.9}$$

from which we obtain

$$\mathcal{D}f = e^T \mathcal{D}^{(0)} f \,, \tag{8.10}$$

where $\mathcal{D}^{(0)}$ is differential operator of standard Courant algebroid (C.4).

We still need to verify that the compatibility conditions in the Courant algebroid definition (6.23) - (6.26) are satisfied for the above choice of anchor, bracket, and differential operator. For the second property (6.23), we have

$$\begin{aligned} [\hat{\Lambda}_1, f\hat{\Lambda}_2]_{\mathcal{C}_\mathcal{T}} &= e^T[e^{-T}\hat{\Lambda}_1, f e^{-T}\hat{\Lambda}_2]_\mathcal{C} \\ &= e^T\Big(f[e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_\mathcal{C} + (\mathcal{L}_{\pi(e^{-T}\hat{\Lambda}_1)} f)(e^{-T}\hat{\Lambda}_2) - \frac{1}{2}\langle e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2\rangle \mathcal{D}^{(0)} f\Big) \\ &= f[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_\mathcal{T}} + (\mathcal{L}_{\rho(\hat{\Lambda}_1)} f)\hat{\Lambda}_2 - \frac{1}{2}\langle \hat{\Lambda}_1, \hat{\Lambda}_2\rangle \mathcal{D} f \,, \end{aligned} \tag{8.11}$$

where we first used the definition of the twisted Courant bracket (8.5), afterward we applied (6.17), and in the end used the expressions for the anchor ρ (8.8) and the differential operator $\mathcal{D}$ (8.10), as well as the fact that $O(D,D)$ transformations keep the inner product invariant.

For the third condition (6.24), we firstly write

$$\begin{aligned}\langle[\hat\Lambda_1,\hat\Lambda_2]_{\mathcal{C}_\mathcal{T}}+\frac{1}{2}\mathcal{D}\langle\hat\Lambda_1,\hat\Lambda_2\rangle,\hat\Lambda_3\rangle &= \langle e^T[e^{-T}\hat\Lambda_1,e^{-T}\hat\Lambda_2]_\mathcal{C}+\frac{1}{2}e^T\mathcal{D}^{(0)}\langle\hat\Lambda_1,\hat\Lambda_2\rangle,\hat\Lambda_3\rangle \\ &= \langle[\Lambda_1,\Lambda_2]_\mathcal{C}+\frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1,\Lambda_2\rangle,\Lambda_3\rangle\,,\end{aligned} \tag{8.12}$$

where we used (8.5), (8.3) and (8.10). Similarly, we obtain

$$\langle\hat\Lambda_2,[\hat\Lambda_1,\hat\Lambda_3]_{\mathcal{C}_\mathcal{T}}+\frac{1}{2}\mathcal{D}\langle\hat\Lambda_1,\hat\Lambda_3\rangle\rangle=\langle\Lambda_2,[\Lambda_1,\Lambda_3]_\mathcal{C}+\frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1,\Lambda_3\rangle\rangle\,, \tag{8.13}$$

and

$$\mathcal{L}_{\rho(\hat\Lambda_1)}\langle\hat\Lambda_2,\hat\Lambda_3\rangle=\mathcal{L}_{\pi(\Lambda_1)}\langle\Lambda_2,\Lambda_3\rangle\,. \tag{8.14}$$

Adding (8.12) and (8.13), we obtain

$$\begin{aligned}&\langle[\hat\Lambda_1,\hat\Lambda_2]_{\mathcal{C}_\mathcal{T}}+\frac{1}{2}\mathcal{D}\langle\hat\Lambda_1,\hat\Lambda_2\rangle,\hat\Lambda_3\rangle+\langle\hat\Lambda_2,[\hat\Lambda_1,\hat\Lambda_3]_{\mathcal{C}_\mathcal{T}}+\frac{1}{2}\mathcal{D}\langle\hat\Lambda_1,\hat\Lambda_3\rangle\rangle= \\ &\langle[\Lambda_1,\Lambda_2]_\mathcal{C}+\frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1,\Lambda_2\rangle,\Lambda_3\rangle+\langle\Lambda_2,[\Lambda_1,\Lambda_3]_\mathcal{C}+\frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1,\Lambda_3\rangle\rangle= \\ &\mathcal{L}_{\pi(\Lambda_1)}\langle\Lambda_2,\Lambda_3\rangle=\mathcal{L}_{\rho(\hat\Lambda_1)}\langle\hat\Lambda_2,\hat\Lambda_3\rangle\,,\end{aligned} \tag{8.15}$$

where in the end we used (C.11) and (8.14).

The fourth condition (6.25) is as easily obtained from the orthogonality of e^T with respect to the inner product

$$\langle\mathcal{D}f,\mathcal{D}g\rangle=\langle e^T\mathcal{D}^{(0)}f,e^T\mathcal{D}^{(0)}g\rangle=\langle\mathcal{D}^{(0)}f,\mathcal{D}^{(0)}g\rangle=0\,. \tag{8.16}$$

Lastly, we note that

$$[[\hat\Lambda_1,\hat\Lambda_2]_{\mathcal{C}_\mathcal{T}},\hat\Lambda_3]_{\mathcal{C}_\mathcal{T}}=e^T[[e^{-T}\hat\Lambda_1,e^{-T}\hat\Lambda_2]_\mathcal{C},e^{-T}\hat\Lambda_3]_\mathcal{C}=e^T[[\Lambda_1,\Lambda_2]_\mathcal{C},\Lambda_3]_\mathcal{C}\,, \tag{8.17}$$

from which we express the Jacobiator (6.19) for the twisted Courant bracket in terms of the Jacobiator for the Courant bracket by

$$\text{Jac}_{\mathcal{C}_\mathcal{T}}(\hat\Lambda_1,\hat\Lambda_2,\hat\Lambda_3)=e^T\text{Jac}_\mathcal{C}(\Lambda_1,\Lambda_2,\Lambda_3)\,. \tag{8.18}$$

Similarly, we note that

$$\langle[\hat\Lambda_1,\hat\Lambda_2]_{\mathcal{C}_\mathcal{T}},\hat\Lambda_3\rangle=\langle e^T[e^{-T}\hat\Lambda_1,e^{-T}\hat\Lambda_2]_\mathcal{C},\hat\Lambda_3\rangle=\langle[\Lambda_1,\Lambda_2]_\mathcal{C},\Lambda_3\rangle\,, \tag{8.19}$$

from which one easily obtains the relation between the Nijenhuis operator (6.20) of the twisted and standard Courant bracket

$$\text{Nij}_{\mathcal{C}_\mathcal{T}}(\hat\Lambda_1,\hat\Lambda_2,\hat\Lambda_3)=\text{Nij}_\mathcal{C}(\Lambda_1,\Lambda_2,\Lambda_3)\,. \tag{8.20}$$

Substituting (8.10), (8.18) and (8.20) into (6.18), we obtain the last compatibility condition of (6.26).

Thus, it has been demonstrated that $\Big(T\mathcal{M} \oplus T^{\star}\mathcal{M}, \langle , \rangle, [,]_{\mathcal{C}_T}, \rho\Big)$ is a Courant algebroid with the appropriate twisted Courant bracket (8.5) as its bracket. This is a simple consequence of the fact that the Courant bracket defines the standard Courant algebroid and that the inner product (6.2) remains invariant under $O(D, D)$ transformations.

It is worth reiterating that the crucial step in obtaining the twisted Courant brackets is a change of basis by the action of $O(D, D)$ transformation. As we will see in the following chapters, it is possible to choose different bases in which the generalized metric is diagonal, and Hamiltonian has the form of a non-interacting Hamiltonian, expressed in terms of non-canonical currents. The Poisson bracket relations of these currents will contain fluxes.

Chapter 9

B-twisted Courant bracket

The procedure for obtaining the twisted Courant algebroid from the Poisson bracket algebra will be applied in this chapter for case of B-transformations. We will obtain the B-twisted Courant bracket, its related Courant algebroid and its Dirac structures.

9.1 Free form Hamiltonian

Consider the background field characterized only with the metric tensor, so that the generalized metric has a simple diagonal form

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & (G^{-1})^{\mu\nu} \end{pmatrix} . \tag{9.1}$$

Acting with the B-transformations $e^{\hat{B}}$ (6.5), we can obtain the usual expression for the generalized metric H_{MN} (2.38)

$$H_{MN} = ((e^{\hat{B}})^T)_M^{\ K}\, G_{KL}\, (e^{\hat{B}})^L_{\ N} . \tag{9.2}$$

Therefore, we can rewrite the canonical Hamiltonian (2.35) in the form of a free Hamiltonian

$$\mathcal{H}_{\mathcal{C}} = \frac{1}{2\kappa}(X^T)^M H_{MN} X^N = \frac{1}{2\kappa}(e^{\hat{B}}X)^M G_{MN} (e^{\hat{B}}X)^N = \frac{1}{2\kappa}\hat{X}^M G_{MN} \hat{X}^N , \tag{9.3}$$

where

$$\hat{X}^M = (e^{\hat{B}})^M_{\ N}\, X^N = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu \end{pmatrix} \equiv \begin{pmatrix} \kappa x'^\mu \\ i_\mu \end{pmatrix} , \tag{9.4}$$

where i_μ is an auxiliary current given by

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu . \tag{9.5}$$

The Poisson bracket algebra of auxiliary currents i_μ is easily obtained with the help of the standard Poisson bracket relations between canonical variables (7.3)

$$\{i_\mu(\sigma), i_\nu(\bar{\sigma})\} = -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma-\bar{\sigma})\,, \tag{9.6}$$

where $B_{\mu\nu\rho}$ is the Kalb-Ramond field strength (2.28). In the context of flux compactification, the Kalb-Ramond field strength is known as the H-flux. Mathematically, this is the exterior derivative of a 2-form B.

Let us now express the generator (7.32) in the non-canonical basis $\hat{X}^M$. It is given by

$$\mathcal{G}^{\hat{B}}_{\hat{\Lambda}} = \int d\sigma \langle \hat{\Lambda}, \hat{X}\rangle = \int d\sigma \Big(\xi^\mu i_\mu + \hat{\lambda}_\mu \kappa x'^\mu\Big)\,, \tag{9.7}$$

which is exactly equal to the generator written in canonical basis when the following relation between gauge parameters is satisfied

$$\hat{\Lambda}^M = (e^{\hat{B}})^M{}_N\,\Lambda^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta_\mu^\nu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \lambda_\mu + 2B_{\mu\nu}\xi^\nu \end{pmatrix} \equiv \begin{pmatrix} \xi^\mu \\ \hat{\lambda}_\mu \end{pmatrix}. \tag{9.8}$$

In the previous chapter, we saw that the $O(D,D)$ transformation on the basis in which the generator is expressed gives rise to the new basis, in which generator closes on the twisted Courant bracket. In this case, we have $e^{\hat{B}}$ as $O(D,D)$ transformation, which when substituted in (8.4) becomes

$$\Big\{\mathcal{G}^{\hat{B}}_{\Lambda_1}, \mathcal{G}^{\hat{B}}_{\Lambda_2}\Big\} = -\mathcal{G}^{\hat{B}}_{[\Lambda_1,\Lambda_2]_{\mathcal{C}_B}}\,, \tag{9.9}$$

where we have marked the B-twisted Courant bracket (8.5) by

$$[\Lambda_1,\Lambda_2]_{\mathcal{C}_B} = e^{\hat{B}}[e^{-\hat{B}}\Lambda_1, e^{-\hat{B}}\Lambda_2]_{\mathcal{C}}\,. \tag{9.10}$$

9.2 B-twisted Courant bracket

We see how the B-twisted Courant bracket can be obtained from the newly defined generator $\mathcal{G}^{\hat{B}}_{\Lambda}$ (9.7). Before that, we require the Poisson bracket relations between the auxiliary currents (9.5) and parameters (9.8), which are easily obtained using the standard Poisson bracket relations

$$\{\xi^\mu(\sigma), i_\nu(\bar{\sigma})\} = \partial_\nu \xi^\mu \delta(\sigma-\bar{\sigma})\,, \quad \{\lambda_\mu(\sigma), i_\nu(\bar{\sigma})\} = \partial_\nu \lambda_\mu \delta(\sigma-\bar{\sigma})\,, \tag{9.11}$$

where we assume the σ dependence unless stated otherwise. We note that the part containing only vector parameters ξ in (9.9) produces additional term containing H-flux, compared to the standard Courant bracket

$$\Big\{\xi_1^\mu(\sigma) i_\mu(\sigma), \xi_2^\nu(\bar{\sigma}) i_\nu(\bar{\sigma})\Big\} = -\Big[\Big(\xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu\Big) i_\mu + 2B_{\mu\nu\rho}\xi_1^\nu \xi_2^\rho \kappa x'^\mu\Big]\delta(\sigma-\bar{\sigma})\,. \tag{9.12}$$

The other contributions remain the same as in the case of the Courant bracket, since

$$\begin{aligned}\Big\{\lambda_{1\mu}(\sigma)\kappa x'^{\mu}(\sigma),\xi_2^{\nu}(\bar\sigma)i_\nu(\bar\sigma)\Big\} &= \Big\{\lambda_{1\mu}(\sigma)\kappa x'^{\mu}(\sigma),\xi_2^{\nu}(\bar\sigma)\pi_\nu(\bar\sigma)\Big\} \\ \Big\{\lambda_{1\mu}(\sigma)\kappa x'^{\mu}(\sigma),\lambda_{2\nu}(\bar\sigma)\kappa x'^{\nu}(\bar\sigma)\Big\} &= 0\,.\end{aligned} \tag{9.13}$$

Substituting (9.12) and (9.13) into (9.9), we obtain the expression for resulting symmetry parameter $\hat\Lambda = \xi \oplus \hat\lambda$ for the B-twisted Courant bracket

$$\begin{aligned}\xi^\mu &= \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu\,, \\ \hat\lambda_\mu &= \xi_1^\nu(\partial_\nu\hat\lambda_{2\mu}-\partial_\mu\hat\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\hat\lambda_{1\mu}-\partial_\mu\hat\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\hat\lambda_2-\xi_2\hat\lambda_1) + 2B_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho\,,\end{aligned} \tag{9.14}$$

or in the coordinate invariant notation

$$\begin{aligned}\xi &= [\xi_1,\xi_2]_L\,, \\ \lambda &= \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) + dB(\xi_1,\xi_2,.)\,.\end{aligned} \tag{9.15}$$

The B-twisted Courant bracket was firstly obtained in [69], where the authors considered a double of a trivial Lie algebroid on cotangent bundle with the bracket $[\lambda_1,\lambda_2]=0$, and a quasi-Lie algebroid whose bracket is defined as

$$[\xi_1,\xi_2]_{\tilde L} = [\xi_1,\xi_2]_L + dB(\xi_1,\xi_2,.)\,. \tag{9.16}$$

Then, the relations (6.11) gives rise to the B-twisted Courant bracket. The bracket (9.16) does not satisfy the Jacobi identity, and for that reason defines a quasi-Lie algebroid. The flux term can be seen as the deformation from the Lie algebroid.

9.3 Courant algebroid

We saw how the B-twisted Courant bracket is directly obtained from the symmetry generator in basis obtained from the appropriate $O(D,D)$ transformation. Substituting $e^T = e^{\hat B}$ into (8.8) and (8.10), we see that the anchor and the derivative operator are defined the same as in the case of non-twisted, standard Courant algebroid, i.e.

$$\rho^{(\hat B)} = \pi\,, \quad \mathcal{D}^{(\hat B)}f = \mathcal{D}^{(0)}f = 0\oplus df\,. \tag{9.17}$$

Let us seek the Dirac structures in the form of $\mathcal{V}_B$ (6.29). On this sub-bundle, the symmetry generator becomes

$$\begin{aligned}\mathcal{G}^{\hat B}_{\mathcal{V}_B(\Lambda)} &= \int d\sigma\Big(\xi^\mu i_\mu + 2B_{\mu\nu}\xi^\nu\kappa x'^\mu\Big) = \int d\sigma\Big(\xi^\mu\pi_\mu + 2B_{\mu\nu}(\xi^\mu\kappa x'^\nu + \xi^\nu\kappa x'^\mu)\Big) \\ &= \int d\sigma\xi^\mu\pi_\mu\,,\end{aligned} \tag{9.18}$$

due to B being antisymmetric. This generator is known to be a generator of diffeomorphisms (7.2), and gives rise to the Lie bracket (7.19) in its Poisson bracket algebra. Hence, $\mathcal{V}_B$ is going to be a Dirac structure no matter what value of the Kalb-Ramond field strength dB, i.e.

$$\Big[\mathcal{V}_B(\Lambda_1), \mathcal{V}_B(\Lambda_2)\Big]_{\mathcal{C}_B} = \mathcal{V}_B\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}_B}\Big)\,, \quad \forall\, dB\,. \tag{9.19}$$

For the standard Courant algebroid, $\mathcal{V}_B$ is a Dirac structure only for a closed 2-form $dB = 0$ (6.34). The twisting of the Courant bracket by B lifted the restriction it imposed on its Dirac structures in the form of $\mathcal{V}_B$.

As for the Dirac structures in the form of $\mathcal{V}_\theta$ (6.31), the restrictions on fluxes remain. The easiest way to see that is to substitute $\xi^\mu = \kappa\theta^{\mu\nu}\lambda_\nu$ into relation (9.14). We obtain

$$\begin{aligned} \xi^\mu &= \kappa^2\theta^{\mu\sigma}\theta^{\nu\rho}(\lambda_{1\rho}\partial_\nu\lambda_{2\sigma} - \lambda_{2\rho}\partial_\nu\lambda_{1\sigma}) + \kappa^2(\theta^{\nu\rho}\partial_\nu\theta^{\mu\sigma} - \theta^{\nu\sigma}\partial_\nu\theta^{\mu\rho})\lambda_{1\rho}\lambda_{2\sigma} \\ \lambda_\mu &= \kappa\theta^{\nu\rho}(\lambda_{1\rho}\partial_\nu\lambda_{2\mu} - \lambda_{2\rho}\partial_\nu\lambda_{1\mu}) + \kappa\partial_\mu\theta^{\nu\rho}\lambda_{1\rho}\lambda_{2\nu} + 2\kappa^2 B_{\mu\nu\rho}\theta^{\nu\alpha}\theta^{\rho\beta}\lambda_{1\alpha}\lambda_{2\beta}\,. \end{aligned} \tag{9.20}$$

For this to define a Dirac structure, the condition $\xi^\mu = \kappa\theta^{\mu\nu}\lambda_\nu$ has to be true on resulting parameters. However, we have instead the relation

$$\xi^\mu = \kappa\theta^{\mu\nu}\lambda_\nu - \kappa^2(\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu} + 2\kappa\theta^{\mu\alpha}\theta^{\nu\beta}\theta^{\rho\gamma}B_{\alpha\beta\gamma})\lambda_{1\nu}\lambda_{2\rho}\,, \tag{9.21}$$

and therefore $\mathcal{V}_\theta$ will be a Dirac structure for

$$[\mathcal{V}_\theta(\Lambda_1), \mathcal{V}_\theta(\Lambda_2)]_{\mathcal{C}_B} = \mathcal{V}_\theta\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}_B}\Big)\,, \quad \mathcal{R} = 0\,, \tag{9.22}$$

where $\mathcal{R}$ is generalized R-flux, given by

$$\mathcal{R}^{\mu\nu\rho} = R^{\mu\nu\rho} + 2\kappa\theta^{\mu\alpha}\theta^{\nu\beta}\theta^{\rho\gamma}B_{\alpha\beta\gamma}\,, \quad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}\,. \tag{9.23}$$

In the coordinate free notation, the generalized $\mathcal{R}$-flux has the expression

$$\mathcal{R} = \frac{1}{2}[\theta, \theta]_S + 2\kappa \wedge^3 \theta\; dB\,, \tag{9.24}$$

where $\wedge^3\theta\; dB$ represents the multiplication of a bi-vector θ three times with the 3-form dB.

The condition $\mathcal{R} = 0$ defines twisted Poisson structures [69]. In that case, one can define the twisted Poisson bracket, using the relation (A.1), which will not satisfy the Jacobi identity. The twisted Poisson structures appeared in many instances in the context of string theory. For instance, they are a suitable mathematical language for describing the non-commutative and non-associative string backgrounds [70]. As we see, in generalized geometry we obtain twisted Poisson structures as Dirac structures of the B-twisted Courant algebroids.

Chapter 10

θ-twisted Courant bracket in symmetry algebra

In this chapter, we will consider the background obtained by the action of θ-transformation, acting on the background characterized solely by the T-dual metric tensor. We will show that Hamiltonian can be written in the diagonal form in a non-canonical basis and that the symmetry generator algebra in that basis closes on θ-twisted Courant bracket. In the end, we will show that this bracket is in fact T-dual to B-twisted Courant bracket.

10.1 Free form Hamiltonian

We will begin with the background characterized solely by the T-dual metric tensor. The T-dual of the diagonal generalized metric G_{MN} (9.1) is given by

$$^\star G_{MN} = \begin{pmatrix} ^\star(G^{-1})_{\mu\nu} & 0 \\ 0 & ^\star G^{\mu\nu} \end{pmatrix} = \begin{pmatrix} G^E_{\mu\nu} & 0 \\ 0 & (G_E^{-1})^{\mu\nu} \end{pmatrix} , \tag{10.1}$$

where the relation (3.21) was used. We will introduce the antisymmetric field with the T-dual of B-transformations, which are θ-transformations $e^{\hat{\theta}}$ (6.7). The T-dual generalized metric becomes

$$\begin{aligned} ^\star H_{MN} &= ((e^{\hat{\theta}})^T)_M^{\ L}\, {}^\star G_{LK} (e^{\hat{\theta}})^K_{\ N} = \begin{pmatrix} 1 & 0 \\ -2^\star B & 1 \end{pmatrix} \begin{pmatrix} ^\star G^{-1} & 0 \\ 0 & ^\star G \end{pmatrix} \begin{pmatrix} 1 & 2^\star B \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} ^\star G^{-1} & 2^\star G^{-1\star} B \\ -2^\star B^\star G^{-1} & ^\star G - 4^\star B^\star G^{-1\star} B \end{pmatrix} = \begin{pmatrix} G_E & 2BG^{-1} \\ -2G^{-1}B & G^{-1} \end{pmatrix} = H_{MN}\,, \end{aligned} \tag{10.2}$$

where in the second line we used (3.21) and (2.38).

We rewrite the Hamiltonian in a non-canonical basis, so that the T-dual generalized metric is diagonal

$$^\star\hat{\mathcal{H}}_C = \frac{1}{2\kappa}(X^T)_M^{\ L}\,((e^{\hat{\theta}})^T)_L^{\ K}\,{}^\star G_{KJ}\,(e^{\hat{\theta}})^J_{\ N}\,X^N = \frac{1}{2\kappa}\hat{X}^M\,{}^\star G_{MN}\hat{X}^N\,, \tag{10.3}$$

where

$$\hat{X}^M = (e^{\hat{\theta}})^M_{\ N}\,X^N = \begin{pmatrix}\delta^\mu_\nu & \kappa\theta^{\mu\nu}\\ 0 & \delta^\nu_\mu\end{pmatrix}\begin{pmatrix}\kappa x'^\nu\\ \pi_\nu\end{pmatrix} = \begin{pmatrix}\kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu\\ \pi_\mu\end{pmatrix} \equiv \begin{pmatrix}k^\mu\\ \pi_\mu\end{pmatrix}\,, \tag{10.4}$$

and where k^μ is an auxiliary current, given by

$$k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu\,. \tag{10.5}$$

The Poisson bracket algebra of these currents is easily obtained from the standard Poisson bracket relations between canonical variables (7.3)

$$\begin{aligned}\{k^\mu(\sigma), k^\nu(\bar{\sigma})\} &= \kappa^2\theta^{\nu\sigma}(\bar{\sigma})\delta^\mu_\sigma\delta'(\sigma-\bar{\sigma}) - \kappa^2\theta^{\mu\rho}\delta^\nu_{\ \rho}\partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma})\\ &\quad+\kappa^2\theta^{\nu\sigma}\partial_\sigma\theta^{\mu\rho}\pi_\rho\delta(\sigma-\bar{\sigma}) - \kappa^2\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho}\pi_\rho\delta(\sigma-\bar{\sigma})\\ &= \kappa^2\theta^{\nu\mu}\delta'(\sigma-\bar{\sigma}) + \kappa^2\partial_\rho\theta^{\nu\mu}x'^\rho\delta(\sigma-\bar{\sigma}) + \kappa^2\theta^{\mu\nu}\delta'(\sigma-\bar{\sigma})\\ &\quad+\kappa^2\theta^{\nu\sigma}\partial_\sigma\theta^{\mu\rho}\pi_\rho\delta(\sigma-\bar{\sigma}) - \kappa^2\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho}\pi_\rho\delta(\sigma-\bar{\sigma})\\ &= -\kappa Q_\rho^{\ \mu\nu}k^\rho\delta(\sigma-\bar{\sigma}) - \kappa^2 R^{\mu\nu\rho}\pi_\rho\delta(\sigma-\bar{\sigma})\,,\end{aligned} \tag{10.6}$$

where in the second step we applied two δ-function identities (7.7) and (7.35), and in the last step we used the inverted relation of (10.5)

$$\kappa x'^\mu = k^\mu - \kappa\theta^{\mu\nu}\pi_\nu\,, \tag{10.7}$$

and expressed the structure coefficients as non-geometric fluxes Q and R, given by

$$Q_\rho^{\ \mu\nu} = \partial_\rho\theta^{\mu\nu}, \qquad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}\,. \tag{10.8}$$

These fluxes can create a potential that stabilizes the vacuum expectation value and provides mass to the moduli [23, 24, 25]. Additionally, Q flux is linked to string non-commutativity [71], while R flux is linked to string non-associativity [72].

The other relevant algebra relation is as easily obtained

$$\{k^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \kappa\delta^\mu_\nu\delta'(\sigma-\bar{\sigma}) + \kappa\partial_\nu\theta^{\mu\rho}\pi_\rho\delta(\sigma-\bar{\sigma})\,. \tag{10.9}$$

We also rewrite the symmetry generator in a new non-canonical basis

$$\mathcal{G}^{\hat{\theta}}_{\hat{\Lambda}} = \int d\sigma\langle\hat{\Lambda}, \hat{X}\rangle\,, \tag{10.10}$$

which is the same as the generator $\mathcal{G}_\Lambda$, when the following relation between the symmetry parameters stand

$$\hat{\Lambda}^M = (e^{\hat{\theta}})^M_{\ N}\, \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu + \kappa\theta^{\mu\nu}\lambda_\nu \\ \lambda_\mu \end{pmatrix} \equiv \begin{pmatrix} \hat{\xi}^\mu \\ \lambda_\mu \end{pmatrix} . \tag{10.11}$$

The algebra of parameters and auxiliary currents is a straightforward application of (7.3), e.g.

$$\{\lambda_\mu(\sigma), k^\nu(\bar{\sigma})\} = \kappa\theta^{\nu\rho}\partial_\rho\lambda_\mu\delta(\sigma-\bar{\sigma})\,, \tag{10.12}$$

and similarly for other cases.

10.2 θ-twisted Courant bracket

Like in the previous chapter, we want to obtain the algebra in the form

$$\left\{\mathcal{G}^{\hat{\theta}}_{\hat{\Lambda}_1},\, \mathcal{G}^{\hat{\theta}}_{\hat{\Lambda}_2}\right\} = -\mathcal{G}^{\hat{\theta}}{}_{[\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathcal{C}_\theta}}\,, \tag{10.13}$$

where $[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_\theta}$ is the θ-twisted Courant bracket. Again, we will do the termwise calculations. From the vector-vector contribution, we obtain the Lie bracket

$$\{\hat{\xi}^\mu_1\pi_\mu(\sigma), \hat{\xi}^\nu_2\pi_\nu(\bar{\sigma})\} = -(\hat{\xi}^\nu_1\partial_\nu\hat{\xi}^\mu_2 - \hat{\xi}^\nu_2\partial_\nu\hat{\xi}^\mu_1)\pi_\mu\delta(\sigma-\bar{\sigma})\,. \tag{10.14}$$

For the form-form bracket, using (10.6) and (10.12) we obtain

$$\begin{aligned} \{\lambda_{1\mu}k^\mu(\sigma), \lambda_{2\nu}k^\nu(\bar{\sigma})\} \;=\; & -\Big(\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu} - \lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}Q_\mu^{\ \rho\nu}\Big)k^\mu\delta(\sigma-\bar{\sigma}) \\ & -\kappa^2 R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\pi_\mu\delta(\sigma-\bar{\sigma})\,. \end{aligned} \tag{10.15}$$

The direct calculations of the form-vector part gives

$$\begin{aligned} \{\lambda_{1\mu}k^\mu(\sigma), \hat{\xi}^\nu_2\pi_\nu(\bar{\sigma})\} \;=\; & \hat{\xi}^\nu_2\partial_\nu\lambda_{1\mu}k^\mu\delta(\sigma-\bar{\sigma}) - \kappa\lambda_{1\mu}\theta^{\mu\rho}\partial_\rho\hat{\xi}^\nu_2\pi_\nu\delta(\sigma-\bar{\sigma}) \\ & +\kappa\lambda_{1\mu}(\sigma)\hat{\xi}^\mu_2(\bar{\sigma})\delta'(\sigma-\bar{\sigma}) + \kappa\lambda_{1\mu}\partial_\nu\theta^{\mu\rho}\pi_\rho\hat{\xi}^\nu_2\delta(\sigma-\bar{\sigma})\,. \end{aligned} \tag{10.16}$$

As in the previous chapters, the anomalous part can be further transformed. Using (7.7) and (7.35), we obtain

$$\begin{aligned} \kappa\lambda_{1\mu}(\sigma)\hat{\xi}^\mu_2(\bar{\sigma})\delta'(\sigma-\bar{\sigma}) \;=\; & \frac{\kappa}{2}\lambda_{1\mu}(\sigma)\hat{\xi}^\mu_2(\bar{\sigma})\Big(\delta'(\sigma-\bar{\sigma}) - \partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma})\Big) \\ =\; & \frac{\kappa}{2}\Big(\lambda_{1\mu}\partial_\nu\hat{\xi}^\mu_2 - \partial_\nu\lambda_{1\mu}\hat{\xi}^\mu_2\Big)x'^\nu\delta(\sigma-\bar{\sigma}) \\ & +\frac{\kappa}{2}\Big(\lambda_{1\mu}(\sigma)\hat{\xi}^\mu_2(\sigma)\delta'(\sigma-\bar{\sigma}) - \lambda_{1\mu}(\bar{\sigma})\hat{\xi}^\mu_2(\bar{\sigma})\partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma})\Big)\,. \end{aligned} \tag{10.17}$$

Now the anomalous part goes to zero after the integration over both σ and $\bar{\sigma}$, while the non-anomalous part can be transformed with the help of (10.7). After relabeling some dummy indices, the expressions for resulting symmetry parameters are

$$\begin{aligned}
\hat{\xi}^\mu &= \hat{\xi}_1^\nu \partial_\nu \hat{\xi}_2^\mu - \hat{\xi}_2^\nu \partial_\nu \hat{\xi}_1^\mu + \\
&+\kappa\theta^{\mu\nu}\Big(\hat{\xi}_1^\rho(\partial_\rho\lambda_{2\nu} - \partial_\nu\lambda_{2\rho}) - \hat{\xi}_2^\rho(\partial_\rho\lambda_{1\nu} - \partial_\nu\lambda_{1\rho}) + \frac{1}{2}\partial_\nu(\hat{\xi}_1\lambda_2 - \hat{\xi}_2\lambda_1)\Big) \\
&+\kappa\hat{\xi}_1^\nu\partial_\nu(\lambda_{2\rho}\theta^{\rho\mu}) - \kappa\hat{\xi}_2^\nu\partial_\nu(\lambda_{1\rho}\theta^{\rho\mu}) + \kappa(\lambda_{1\nu}\theta^{\nu\rho})\partial_\rho\hat{\xi}_2^\mu - \kappa(\lambda_{2\nu}\theta^{\nu\rho})\partial_\rho\hat{\xi}_1^\mu \\
&+\kappa^2 R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,, \\
\lambda_\mu &= \hat{\xi}_1^\nu(\partial_\nu\lambda_{2\mu} - \partial_\mu\lambda_{2\nu}) - \hat{\xi}_2^\nu(\partial_\nu\lambda_{1\mu} - \partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\hat{\xi}_1\lambda_2 - \hat{\xi}_2\lambda_1) \\
&+\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu} - \lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}Q_\mu^{\ \rho\nu}\,.
\end{aligned} \tag{10.18}$$

In the coordinate free notation, the above expressions read

$$\begin{aligned}
\hat{\xi} =&[\hat{\xi}_1, \hat{\xi}_2]_L - \kappa[\hat{\xi}_2, \lambda_1\theta]_L + \kappa[\hat{\xi}_1, \lambda_2\theta]_L + \frac{\kappa^2}{2}[\theta,\theta]_S(\lambda_1,\lambda_2,.) \\
&- \kappa\theta\Big(\mathcal{L}_{\hat{\xi}_2}\lambda_1 - \mathcal{L}_{\hat{\xi}_1}\lambda_2 + \frac{1}{2}d(i_{\hat{\xi}_1}\lambda_2 - i_{\hat{\xi}_2}\lambda_1)\Big) \\
\lambda =&\mathcal{L}_{\hat{\xi}_1}\lambda_2 - \mathcal{L}_{\hat{\xi}_2}\lambda_1 - \frac{1}{2}d(i_{\hat{\xi}_1}\lambda_2 - i_{\hat{\xi}_2}\lambda_1) + \kappa[\lambda_1,\lambda_2]_\theta\,,
\end{aligned} \tag{10.19}$$

where $[\theta,\theta]_S$ represents the Schouten-Nijenhuis bracket (4.25) [47], and $[\lambda_1,\lambda_2]_\theta$ is the Koszul bracket (5.11) [53], and $\theta(\lambda)$ is defined as in (5.9).

10.3 Courant algebroid

The obtained bracket is the θ-twisted Courant bracket, given by

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_\theta} = e^{\hat{\theta}}[e^{-\hat{\theta}}\Lambda_1, e^{-\hat{\theta}}\Lambda_2]_{\mathcal{C}}\,. \tag{10.20}$$

It defines the Courant algebroid with the following anchor

$$\rho^{(\hat{\theta})}(\Lambda) = \xi^\mu - \kappa\theta^{\mu\nu}\lambda_\nu\,, \tag{10.21}$$

from which we easily obtain the differential operator

$$D^{(\hat{\theta})}f = \begin{pmatrix} \kappa\theta^{\mu\nu}\partial_\nu f \\ \partial_\mu f \end{pmatrix} = \begin{pmatrix} d_\theta f \\ df \end{pmatrix}\,, \tag{10.22}$$

where d_θ (5.20) is the exterior derivative corresponding to the Koszul bracket. The structure $\Big(T\mathcal{M}\oplus T^\star\mathcal{M}, [,]_{\mathcal{C}_\theta}, \langle,\rangle, \rho^{(\theta)}\Big)$ is a Courant algebroid, with all compatibility conditions satisfied.

To obtain the Dirac structures, we firstly substitute the graph of a 2-form into the definition of the symmetry generator (10.10)

$$\begin{aligned} \mathcal{G}^{\hat\theta}_{\mathcal{V}_B(\Lambda)} &= \int d\sigma\Big(\hat\xi^\mu\pi_\mu + 2B_{\mu\nu}\hat\xi^\nu\kappa\theta^{\mu\rho}\pi_\rho + 2B_{\mu\nu}\kappa x'^\mu\hat\xi^\nu\Big) \\ &= \int d\sigma\Big(\hat\xi^\nu(\delta^\mu_\nu - 2\kappa B_{\nu\rho}\theta^{\rho\mu})\pi_\mu + 2B_{\mu\nu}\kappa x'^\nu\hat\xi^\nu\Big)\,. \end{aligned} \tag{10.23}$$

Next, we can use the following identity

$$\delta^\mu_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu} + (G_E^{-1})^{\mu\rho}G_{\rho\nu}\,, \tag{10.24}$$

which is easily obtained from (2.36) and (3.16). The generator (10.23) can be rewritten in canonical form as

$$\mathcal{G}^{\hat\theta}_{\mathcal{V}_B(\Lambda)} = \int d\sigma\Big(\tilde\xi^\mu\pi_\mu + \tilde\lambda_\mu\kappa x'^\mu\Big) = \mathcal{G}_{\tilde\Lambda}\,,\quad \tilde\xi^\mu = \hat\xi^\mu - 2\kappa\theta^{\mu\nu}B_{\nu\rho}\hat\xi^\rho\,,\ \tilde\lambda_\mu = 2(BG^{-1}G_E)_{\mu\nu}\tilde\xi^\nu\,. \tag{10.25}$$

The generator written like this will give rise to the Courant bracket. We can use the results from previous chapters, namely (6.34), to establish that the condition for $\mathcal{V}_B$ to be Dirac structure is given by

$$d(BG^{-1}G_E) = dB - 4d(BG^{-1}BG^{-1}B) = 0\,. \tag{10.26}$$

On the other hand, when we substitute the graph of the bi-vector into (10.10), the generator becomes

$$\mathcal{G}^{\hat\theta}_{\mathcal{V}_\theta(\Lambda)} = \int d\sigma\kappa x'^\mu\lambda_\mu\,, \tag{10.27}$$

which does not depend on canonical momenta, and hence $\mathcal{V}_\theta$ is always Dirac structure. Moreover, we have

$$[\mathcal{V}_\theta(\Lambda_1), \mathcal{V}_\theta(\Lambda_2)]_{\mathcal{C}_\theta} = 0\,. \tag{10.28}$$

This reflects the basic asymmetry of the Courant bracket in the way how it treats vectors and 1-forms. On vectors, it reduces to the Lie bracket, which happens when the B-twisted Courant bracket is considered on the subspace $\mathcal{V}_B$. On 1-forms, the Courant bracket is zero. The θ-twisted Courant bracket becomes precisely the Courant bracket of 1-forms on $\mathcal{V}_\theta$.

10.4 Relation to B-twisted Courant bracket via self T-duality

Suppose we want to implement T-duality within the same phase space, without adding any new D coordinates or their corresponding momenta. Such a transformation should swap the momenta π_μ

with the coordinate σ-derivatives, as well as the background fields with their T-dual counterparts, i.e.

$$\pi_\mu \leftrightarrow \kappa x'^\mu \,, \quad 2B_{\mu\nu} \leftrightarrow \kappa\theta^{\mu\nu} \,. \tag{10.29}$$

We will use the term "self T-duality" to describe this concept. Under such transformation, auxiliary currents i_μ (9.5) and k^μ (10.5) transform one into another

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu \leftrightarrow \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu = k^\mu \,. \tag{10.30}$$

We can conclude that the generators (9.7) and (10.10) are also related by self T-duality, and so is their algebra. This means that B-twisted Courant bracket and θ-twisted Courant bracket are mutually related by self T-duality. The Courant algebroid with H-flux under the exchange of mutually T-dual variables becomes the Courant algebroid with Q and R fluxes.

The advantage of considering the T-duality and all generators in the same phase space is that we can easily express one in terms of the other by coordinate transformation. Inverting relations (9.8) and substituting it into (10.11), we obtain the following relation between the parameters

$$\hat{\Lambda}^M_{(\theta)} = (e^{\hat{\theta}} e^{-\hat{B}})^M_{\ N}\, \hat{\Lambda}^N_{(B)} \,, \tag{10.31}$$

where in order to differentiate between parameters (9.8) and (10.11), we added indices B and θ. We have obtained the isomorphism between these Courant algebroids

$$\varphi = (e^{\hat{\theta}} e^{-\hat{B}})^M_{\ N} = \begin{pmatrix} \delta^\mu_\nu - 2\kappa(\theta B)^\mu_{\ \nu} & \kappa\theta^{\mu\nu} \\ -2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \,. \tag{10.32}$$

The isomorphism φ satisfies the first rule of (6.27), simply from the fact that the inner product is invariant under the $O(D,D)$ transformations. The second property can be also easily verified. Firstly, from relation (9.10) we can derive

$$\varphi[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = e^{\hat{\theta}}[e^{-\hat{B}}\Lambda_1, e^{-\hat{B}}\Lambda_2]_{\mathcal{C}} \,, \tag{10.33}$$

and from (10.20) we obtain

$$[\varphi(\Lambda_1), \varphi(\Lambda_2)]_{\mathcal{C}_\theta} = e^{\hat{\theta}}[e^{-\hat{B}}\Lambda_1, e^{-\hat{B}}\Lambda_2]_{\mathcal{C}} \,, \tag{10.34}$$

and therefore we obtain the second requirement in (6.27)

$$\varphi[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = \Big([\varphi(\Lambda_1), \varphi(\Lambda_2)]_{\mathcal{C}_\theta}\Big) \,. \tag{10.35}$$

We demonstrated that the Courant algebroid relations that govern symmetry generator algebra in both the initial and self T-dual picture are isomorphic. This isomorphism is governed by the same properties (6.27) as the isomorphism that governs topological T-duality on backgrounds defined on tori [63]. As a result, we extended the idea of T-duality as a Courant algebroid isomorphism to include symmetry transformations as well.

Chapter 11

B-θ twisted Courant bracket

The focus of this chapter is on creating a Courant bracket that is twisted by both B and θ fields. The first step is to construct a twisting matrix that includes both fields and is self T-dual. Next, we derive the necessary algebraic relations to obtain the complete bracket, along with all the corresponding generalized fluxes. We then explain how these fluxes can be interpreted in terms of deformations of Lie algebroids.

11.1 Twisting transformation

The B-twisted Courant bracket is obtained with the action of $e^{\hat{B}}$, while the θ-twisted Courant bracket is obtained with the action of $e^{\hat{\theta}}$ transformation on the basis of a double generator. Initially, one might think to obtain the twist by B and θ by using the product of transformations $e^{\hat{B}}$ and $e^{\hat{\theta}}$. This would indeed give rise to the twisted Courant bracket since the composition of two group elements remains in that group. However, the two transformations do not commute, i.e.

$$e^{\hat{B}}e^{\hat{\theta}} = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \kappa\theta \\ 2B & 1+2\kappa B\theta \end{pmatrix} , \tag{11.1}$$

$$e^{\hat{\theta}}e^{\hat{B}} = \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix} = \begin{pmatrix} 1+2\kappa\theta B & \kappa\theta \\ 2B & 1 \end{pmatrix} . \tag{11.2}$$

The Courant bracket twisted with the transformation (11.1) yields the Roytenberg bracket. This is an extension of the Courant bracket which includes all fluxes and has been derived several times [60, 73, 74]. However, it is unclear why that bracket is to be preferred to the Courant bracket twisted with the transformation (11.2). Moreover, neither of the brackets is invariant under T-duality. Our goal is to twist the bracket in a way that treats B and θ equally and maintains the T-dual invariance of the bracket. This we will refer to as the simultaneous twisting of the Courant bracket by both B and θ [3]. Let us define $\breve{B}$ by

$$\breve{B} = \hat{B} + \hat{\theta} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 2B_{\mu\nu} & 0 \end{pmatrix} . \tag{11.3}$$

By construction, $\breve{B}$ is invariant under self T-duality and it treats the Kalb-Ramond field B and the non-commutativity parameter θ equally. Therefore, we defined the Courant bracket twisted at the same time by a 2-form B and by a bi-vector θ by

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_{\breve{B}}} = e^{\breve{B}}[e^{-\breve{B}}\Lambda_1, e^{-\breve{B}}\Lambda_2]_{\mathcal{C}} . \tag{11.4}$$

It is not as straightforward to derive the formula for the matrix $e^{\breve{B}}$ as it was in the previous cases. The reason for this is that while the matrices $\hat{B}$ and $\hat{\theta}$ have a squared value of zero, the same is not true for the matrix $\breve{B}$. Hence, the full Taylor expression

$$e^{\breve{B}} = \sum_{n=0}^{\infty} \frac{\breve{B}^n}{n!} \tag{11.5}$$

has to be obtained. The square of the matrix $\breve{B}$ is given by

$$\breve{B}^2 = \begin{pmatrix} 2\kappa(\theta B)^\mu{}_\nu & 0 \\ 0 & 2\kappa(B\theta)_\mu{}^\nu \end{pmatrix} = \begin{pmatrix} \alpha^\mu{}_\nu & 0 \\ 0 & (\alpha^T)_\mu{}^\nu \end{pmatrix} , \tag{11.6}$$

while its cube is given by

$$\breve{B}^3 = \begin{pmatrix} 0 & 2\kappa^2(\theta B\theta)^{\mu\nu} \\ 4\kappa(B\theta B)_{\mu\nu} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \kappa\alpha^\mu{}_\rho\theta^{\rho\nu} \\ 2B_{\mu\rho}\alpha^\rho{}_\nu & 0 \end{pmatrix} , \tag{11.7}$$

where we have marked

$$\alpha^\mu{}_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu} . \tag{11.8}$$

The matrix α is defined for pure convenience, and it possesses a couple of useful properties. Firstly, it is a symmetric matrix

$$\alpha^\mu{}_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu} = 2\kappa(-\theta^{\rho\mu})(-B_{\nu\rho}) = 2\kappa B_{\nu\rho}\theta^{\rho\mu} = (\alpha^T)_\nu{}^\mu , \tag{11.9}$$

and it transforms into its transpose under the self T-duality relations (10.29)

$$\alpha^\mu{}_\nu \leftrightarrow (\alpha^T)_\nu{}^\mu . \tag{11.10}$$

Moreover, we have

$$\begin{aligned}\theta^{\mu\rho}(\alpha^T)_\rho{}^\nu &= 2\kappa\theta^{\mu\rho}B_{\rho\sigma}\theta^{\sigma\nu} = \alpha^\mu{}_\sigma\theta^{\sigma\nu} \\ B_{\mu\rho}\alpha^\rho{}_\nu &= 2\kappa B_{\mu\rho}\theta^{\rho\sigma}B_{\sigma\nu} = (\alpha^T)_\mu{}^\sigma B_{\sigma\nu}\,.\end{aligned} \tag{11.11}$$

One easily observes the regularity that degrees of $\breve{B}$ possess. The even degrees are given by

$$\breve{B}^{2n} = \begin{pmatrix}(\alpha^n)^\mu{}_\nu & 0 \\ 0 & ((\alpha^T)^n)_\mu{}^\nu\end{pmatrix}\,, \tag{11.12}$$

while the odd degrees by

$$\breve{B}^{2n+1} = \begin{pmatrix}0 & \kappa(\alpha^n\theta)^{\mu\nu} \\ 2(B\alpha^n)_{\mu\nu} & 0\end{pmatrix}\,, \tag{11.13}$$

where we applied (11.11). We can now substitute (11.12) and (11.13) into (11.5), and write

$$e^{\breve{B}} = \begin{pmatrix}\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n)!}\right)^\mu{}_\nu & \kappa\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\right)^\mu{}_\rho\theta^{\rho\nu} \\ 2B_{\mu\rho}\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\right)^\rho{}_\nu & \left(\sum_{n=0}^\infty \frac{(\alpha^T)^n}{(2n)!}\right)_\mu{}^\nu\end{pmatrix}\,. \tag{11.14}$$

The terms in the twisting matrix can be simplified using the Taylor expressions for hyperbolic functions

$$\cosh\alpha = \sum_{n=0}^\infty \frac{\alpha^{2n}}{(2n)!}\,, \quad \sinh\alpha = \sum_{n=0}^\infty \frac{\alpha^{2n+1}}{(2n+1)!}\,, \tag{11.15}$$

from which we derive

$$\cosh\sqrt{\alpha} = \sum_{n=0}^\infty \frac{\alpha^n}{(2n)!}\,, \quad \frac{\sinh\sqrt{\alpha}}{\sqrt{\alpha}} = \sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\,. \tag{11.16}$$

For our convenience let us introduce $\mathcal{S}^\mu{}_\nu$ and $\mathcal{C}^\mu{}_\nu$ by

$$\mathcal{S}^\mu{}_\nu = \left(\frac{\sinh\sqrt{\alpha}}{\sqrt{\alpha}}\right)^\mu{}_\nu\,, \quad \mathcal{C}^\mu{}_\nu = \left(\cosh\sqrt{\alpha}\right)^\mu{}_\nu\,. \tag{11.17}$$

We can now rewrite the full transformation matrix $e^{\breve{B}}$ as

$$e^{\breve{B}} = \begin{pmatrix}\mathcal{C}^\mu{}_\nu & \kappa\mathcal{S}^\mu{}_\rho\theta^{\rho\nu} \\ 2B_{\mu\rho}\mathcal{S}^\rho{}_\nu & (\mathcal{C}^T)_\mu{}^\nu\end{pmatrix}\,. \tag{11.18}$$

Due to (11.9), the hyperbolic functions (11.17) are symmetric

$$\mathcal{S}^\mu{}_\nu = (\mathcal{S}^T)_\nu{}^\mu\,, \quad \mathcal{C}^\mu{}_\nu = (\mathcal{C}^T)_\nu{}^\mu\,. \tag{11.19}$$

Secondly, the relation (11.11) is easily generalized to higher orders of α, from which we obtain

$$\begin{aligned} \mathcal{S}^\mu{}_\rho \theta^{\rho\nu} &= \theta^{\mu\rho} (\mathcal{S}^T)_\rho{}^\nu \,, \quad \mathcal{C}^\mu{}_\rho \theta^{\rho\nu} = \theta^{\mu\rho} (\mathcal{C}^T)_\rho{}^\nu \,, \\ B_{\mu\rho} \mathcal{S}^\rho{}_\nu &= (\mathcal{S}^T)_\mu{}^\rho B_{\rho\nu} \,, \quad B_{\mu\rho} \mathcal{C}^\rho{}_\nu = (\mathcal{C}^T)_\mu{}^\rho B_{\rho\nu} \,. \end{aligned} \tag{11.20}$$

Thirdly, the well-known hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$ can also be expressed in terms of newly defined tensors by

$$(\mathcal{C}^2)^\mu{}_\nu - \alpha^\mu{}_\rho (\mathcal{S}^2)^\rho{}_\nu = \delta^\mu_\nu \,. \tag{11.21}$$

Lastly, from (11.10) we conclude

$$\mathcal{C} \leftrightarrow \mathcal{C}^T \,, \;\; \mathcal{S} \leftrightarrow \mathcal{S}^T \,. \tag{11.22}$$

The transformation has been obtained, but we need to check whether it is an $O(D, D)$ transformation, and therefore suitable for twisting the Courant bracket and defining the Courant algebroid. The transpose of $e^{\breve{B}}$ is given by

$$(e^{\breve{B}})^T = \begin{pmatrix} (\mathcal{C}^T)_\nu{}^\mu & -2B_{\mu\rho}\mathcal{S}^\rho{}_\nu \\ -\kappa \mathcal{S}^\mu{}_\rho \theta^{\rho\nu} & \mathcal{C}^\mu{}_\nu \end{pmatrix} , \tag{11.23}$$

and therefore it can be easily verified that

$$\begin{aligned} (e^{\breve{B}})^T \, \eta \, e^{\breve{B}} &= \begin{pmatrix} \mathcal{C}^T & -2B\mathcal{S} \\ -\kappa\mathcal{S}\theta & \mathcal{C} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{C} & \kappa\mathcal{S}\theta \\ 2B\mathcal{S} & \mathcal{C}^T \end{pmatrix} \\ &= \begin{pmatrix} 2\mathcal{C}^T B\mathcal{S} - 2B\mathcal{S}\mathcal{C} & (\mathcal{C}^T)^2 - 2\kappa B\mathcal{S}^2\theta \\ -2\kappa\mathcal{S}\theta B\mathcal{S} + \mathcal{C}^2 & -\kappa\mathcal{S}\theta\mathcal{C}^T + \kappa\mathcal{C}\mathcal{S}\theta \end{pmatrix} = \eta \,, \end{aligned} \tag{11.24}$$

where η is $O(D, D)$ invariant metric (6.4), and in the last step the properties (11.20) and (11.21) were used. The determinant of $e^{\breve{B}}$ is given by

$$\det(e^{\breve{B}}) = e^{Tr(\breve{B})} = 1 \,, \tag{11.25}$$

and its inverse by

$$e^{-\breve{B}} = \begin{pmatrix} \mathcal{C}^\mu{}_\nu & -\kappa\mathcal{S}^\mu{}_\rho \theta^{\rho\nu} \\ -2B_{\mu\rho}\mathcal{S}^\rho{}_\nu & (\mathcal{C}^T)_\mu{}^\nu \end{pmatrix} , \tag{11.26}$$

which is in the accordance with (B.4).

11.2 Self T-dual generator

Having constructed the $O(D, D)$-invariant self T-dual twisting transformation with both B and θ, we can move forward with the approach outlined in Chapter 8 to derive the corresponding twisted Courant bracket using the generator algebra. The Poisson bracket representation of the Courant bracket twisted by B and θ can be obtained from the generator written in the basis

$$\breve{X}^M = (e^{\breve{B}})^M_{\ N}\, X^N = \begin{pmatrix} \breve{k}^\mu \\ \breve{\iota}_\mu \end{pmatrix} , \tag{11.27}$$

where the new currents $\breve{\iota}_\mu$ and $\breve{k}^\mu$ are given by

$$\begin{aligned} \breve{k}^\mu &= \kappa \mathcal{C}^\mu_{\ \nu} x'^\nu + \kappa (\mathcal{S}\theta)^{\mu\nu} \pi_\nu \, , \\ \breve{\iota}_\mu &= 2(B\mathcal{S})_{\mu\nu} x'^\nu + (\mathcal{C}^T)_\mu^{\ \nu} \pi_\nu \, . \end{aligned} \tag{11.28}$$

These currents are mutually related by self T-duality (10.29), implying that the generator is invariant under self T-duality. Using (11.26), the relations (11.28) can easily be inverted

$$\begin{aligned} \kappa x'^\mu &= \mathcal{C}^\mu_{\ \nu} \breve{k}^\nu - \kappa (\mathcal{S}\theta)^{\mu\nu} \breve{\iota}_\nu \, , \\ \pi_\mu &= -2(B\mathcal{S})_{\mu\nu} \breve{k}^\nu + (\mathcal{C}^T)_\mu^{\ \nu} \breve{\iota}_\nu \, . \end{aligned} \tag{11.29}$$

The double generator is given by

$$\breve{\mathcal{G}}_{\breve{\Lambda}} = \int d\sigma \langle \breve{X}, \breve{\Lambda} \rangle \, , \tag{11.30}$$

and is equal to the generator $\mathcal{G}_\Lambda$ (7.33) for

$$\breve{\Lambda}^M = (e^{\breve{B}})^M_{\ N}\, \Lambda^N = \begin{pmatrix} \breve{\xi}^\mu \\ \breve{\lambda}_\mu \end{pmatrix} , \tag{11.31}$$

where

$$\begin{aligned} \breve{\xi}^\mu &= \mathcal{C}^\mu_{\ \nu} \xi^\nu + \kappa (\mathcal{S}\theta)^{\mu\nu} \lambda_\nu \, , \\ \breve{\lambda}_\mu &= 2(B\mathcal{S})_{\mu\nu} \xi^\nu + (\mathcal{C}^T)_\mu^{\ \nu} \lambda_\nu \, . \end{aligned} \tag{11.32}$$

We could proceed by directly computing the Poisson bracket between the generators $\breve{\mathcal{G}}_{\breve{\Lambda}}$. However, the interpretation of terms is easier if an auxiliary basis is introduced by

$$\mathring{X}^M = \begin{pmatrix} \mathcal{C}^\mu_{\ \nu} & 0 \\ 0 & ((\mathcal{C}^{-1})^T)_\mu^{\ \nu} \end{pmatrix} \begin{pmatrix} \breve{k}^\nu \\ \breve{\iota}_\nu \end{pmatrix} = \begin{pmatrix} \mathring{k}^\mu \\ \mathring{\iota}_\mu \end{pmatrix} , \tag{11.33}$$

where the auxiliary currents are

$$\begin{aligned} \mathring{k}^\mu &= \kappa x'^\mu + \kappa \mathring{\theta}^{\mu\nu} \mathring{\iota}_\nu \, , \\ \mathring{\iota}_\mu &= \pi_\mu + 2\kappa \mathring{B}_{\mu\nu} x'^\nu \, , \end{aligned} \tag{11.34}$$

and auxiliary fields are

$$\mathring{B}_{\mu\nu} = B_{\mu\rho}\mathcal{S}^{\rho}_{\ \sigma}(\mathcal{C}^{-1})^{\sigma}_{\ \nu}\,, \tag{11.35}$$

and

$$\mathring{\theta}^{\mu\nu} = \mathcal{C}^{\mu}_{\ \rho}\mathcal{S}^{\rho}_{\ \sigma}\theta^{\sigma\nu}\,. \tag{11.36}$$

The hyperbolic functions of the background fields are incorporated into the auxiliary background fields in the auxiliary basis. Additionally, the auxiliary currents (11.34) take the same form as the currents that generate the Roytenberg bracket [60], although they depend on a different set of fields. We can easily invert the relation (11.33), and obtain

$$\breve{k}^{\mu} = \mathcal{C}^{\mu}_{\ \nu}\mathring{k}^{\mu}\,, \quad \breve{\imath}_{\mu} = \mathring{\imath}_{\nu}\mathcal{C}^{\nu}_{\ \mu}\,. \tag{11.37}$$

Moreover, the coordinate σ-derivative is as easily expressed in terms of auxiliary currents by

$$\kappa x'^{\mu} = \mathring{k}^{\mu} - \kappa\mathring{\theta}^{\mu\nu}\mathring{\imath}_{\nu}\,, \tag{11.38}$$

all of which simplifies computations substantially.

We will find the algebra of the auxiliary currents with auxiliary fluxes as its structure functions. Using these relations and equation (11.37), we will then determine the necessary algebraic relations for the twisted Courant bracket with both B and θ backgrounds.

11.3 Algebra of auxiliary currents

The Poisson bracket algebra of the auxiliary currents $\mathring{\imath}_{\mu}$ is given by

$$\{\mathring{\imath}_{\mu}(\sigma), \mathring{\imath}_{\nu}(\bar{\sigma})\} = -2\mathring{B}_{\mu\nu\rho}\mathring{k}^{\rho}\delta(\sigma-\bar{\sigma}) - \mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho}\,\mathring{\imath}_{\rho}\delta(\sigma-\bar{\sigma})\,, \tag{11.39}$$

where

$$\mathring{B}_{\mu\nu\rho} = \partial_{\mu}\mathring{B}_{\nu\rho} + \partial_{\nu}\mathring{B}_{\rho\mu} + \partial_{\rho}\mathring{B}_{\mu\nu}\,, \tag{11.40}$$

and

$$\mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho} = -2\kappa\mathring{B}_{\mu\nu\sigma}\mathring{\theta}^{\sigma\rho}\,. \tag{11.41}$$

The algebra of currents $\mathring{k}^{\mu}$ is given by

$$\{\mathring{k}^{\mu}(\sigma), \mathring{k}^{\nu}(\bar{\sigma})\} = -\kappa\mathring{\mathcal{Q}}_{\rho}^{\ \mu\nu}\mathring{k}^{\rho}\delta(\sigma-\bar{\sigma}) - \kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\imath}_{\rho}\delta(\sigma-\bar{\sigma})\,, \tag{11.42}$$

where

$$\mathring{\mathcal{Q}}_{\mu}^{\ \nu\rho} = \mathring{Q}_{\mu}^{\ \nu\rho} + 2\kappa\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{B}_{\mu\sigma\tau}\,, \quad \mathring{Q}_{\mu}^{\ \nu\rho} = \partial_{\mu}\mathring{\theta}^{\nu\rho}\,, \tag{11.43}$$

and

$$\mathring{\mathcal{R}}^{\mu\nu\rho} = \mathring{R}^{\mu\nu\rho} + 2\kappa\mathring{\theta}^{\mu\lambda}\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{B}_{\lambda\sigma\tau}\,, \quad \mathring{R}^{\mu\nu\rho} = \mathring{\theta}^{\mu\sigma}\partial_\sigma\mathring{\theta}^{\nu\rho} + \mathring{\theta}^{\nu\sigma}\partial_\sigma\mathring{\theta}^{\rho\mu} + \mathring{\theta}^{\rho\sigma}\partial_\sigma\mathring{\theta}^{\mu\nu}\,. \tag{11.44}$$

The remaining algebra of currents $\mathring{k}^\mu$ and $\mathring{\imath}_\mu$ can be as easily obtained

$$\{\mathring{\imath}_\mu(\sigma), \mathring{k}^\nu(\bar{\sigma})\} = \kappa\delta_\mu^\nu\delta'(\sigma-\bar{\sigma}) + \mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu}\,\mathring{k}^\rho\delta(\sigma-\bar{\sigma}) - \kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho}\mathring{\imath}_\rho\delta(\sigma-\bar{\sigma})\,. \tag{11.45}$$

These algebra relations can be summarized in the double formalism as

$$\{\mathring{X}^M, \mathring{X}^N\} = -\mathring{F}^{MN}_{\ \ \ \ P}\,\mathring{X}^P\delta(\sigma-\bar{\sigma}) + \kappa\eta^{MN}\delta'(\sigma-\bar{\sigma})\,, \tag{11.46}$$

with

$$F^{MN\rho} = \begin{pmatrix} \kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho} & -\kappa\mathring{\mathcal{Q}}_\nu^{\ \mu\rho} \\ \kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho} & \mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho} \end{pmatrix}\,, \qquad F^{MN}{}_\rho = \begin{pmatrix} \kappa\mathring{\mathcal{Q}}_\rho^{\ \mu\nu} & \mathring{\mathcal{F}}_{\nu\rho}^{\ \ \mu} \\ -\mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu} & 2\mathring{B}_{\mu\nu\rho} \end{pmatrix}\,. \tag{11.47}$$

The terms appearing are generalized fluxes [20, 21, 22]. Now we can use these relations to obtain the fluxes related to the Courant bracket twisted with B and θ.

11.4 Fluxes of self T-dual currents

The algebra of auxiliary currents closes with auxiliary fluxes as its structure coefficients. We can now use the expression for self T-dual currents $\breve{k}$ and $\breve{\imath}$ in terms of their auxiliary counterparts (11.37) to compute the fluxes relevant for the Courant bracket twisted with both B and θ. Besides the relations (11.46), we will require the algebra relations in the form

$$\{\mathcal{C}^\mu_{\ \rho}(\sigma), \mathring{\imath}_\nu(\bar{\sigma})\} = \partial_\nu\mathcal{C}^\mu_{\ \rho}\delta(\sigma-\bar{\sigma})\,, \quad \{\mathcal{C}^\mu_{\ \rho}(\sigma), \mathring{k}^\nu(\bar{\sigma})\} = \kappa\mathring{\theta}^{\nu\sigma}\partial_\sigma\mathcal{C}^\mu_{\ \rho}\delta(\sigma-\bar{\sigma})\,, \tag{11.48}$$

and similar identities for algebra between variables that do not depend on momenta with currents that form the self T-dual basis.

Firstly, we write

$$\begin{aligned}
\{\breve{\imath}_\mu(\sigma), \breve{\imath}_\nu(\bar{\sigma})\} &= \mathcal{C}^\rho_{\ \mu}\mathcal{C}^\sigma_{\ \nu}\{\mathring{\imath}_\rho, \mathring{\imath}_\sigma\} + \mathcal{C}^\rho_{\ \mu}\mathring{\imath}_\sigma\{\mathring{\imath}_\rho, \mathcal{C}^\sigma_{\ \nu}\} + \mathring{\imath}_\rho\mathcal{C}^\sigma_{\ \nu}\{\mathcal{C}^\rho_{\ \mu}, \mathring{\imath}_\sigma\} \\
&= -2\mathcal{C}^\rho_{\ \mu}\mathcal{C}^\sigma_{\ \nu}\mathring{B}_{\rho\sigma\alpha}\mathring{k}^\alpha\delta(\sigma-\bar{\sigma}) - \mathcal{C}^\rho_{\ \mu}\mathcal{C}^\sigma_{\ \nu}\mathring{\mathcal{F}}_{\rho\sigma}^{\ \ \alpha}\,\mathring{\imath}_\alpha\delta(\sigma-\bar{\sigma}) \\
&\quad -\Big(\mathcal{C}^\rho_{\ \mu}\partial_\rho\mathcal{C}^\sigma_{\ \nu} - \mathcal{C}^\rho_{\ \nu}\partial_\rho\mathcal{C}^\sigma_{\ \mu}\Big)\mathring{\imath}_\sigma\delta(\sigma-\bar{\sigma}) \\
&= -2\breve{\mathcal{B}}_{\mu\nu\rho}\breve{k}^\rho\delta(\sigma-\bar{\sigma}) - \breve{\mathcal{F}}_{\mu\nu}^{\ \ \rho}\,\breve{\imath}_\rho\delta(\sigma-\bar{\sigma})\,,
\end{aligned} \tag{11.49}$$

where we firstly used (11.37), after which we substituted (11.39) and (11.48). The resulting algebra we expressed in terms of $\breve{\mathcal{B}}$ flux, given by

$$\breve{\mathcal{B}}_{\mu\nu\rho} = \mathcal{C}^\alpha_{\ \mu}\mathcal{C}^\beta_{\ \nu}\mathcal{C}^\gamma_{\ \rho}\mathring{B}_{\alpha\beta\gamma}\,, \tag{11.50}$$

and $\breve{\mathcal{F}}$ flux, given by

$$\breve{\mathcal{F}}_{\mu\nu}^{\ \rho} = \mathring{\mathcal{F}}_{\alpha\beta}^{\ \gamma}\mathcal{C}^\alpha_{\ \mu}\mathcal{C}^\beta_{\ \nu}(\mathcal{C}^{-1})^\rho_{\ \gamma} + (\mathcal{C}^{-1})^\rho_{\ \tau}\Big(\mathcal{C}^\sigma_{\ \mu}\partial_\sigma\mathcal{C}^\tau_{\ \nu} - \mathcal{C}^\sigma_{\ \nu}\partial_\sigma\mathcal{C}^\tau_{\ \mu}\Big)\,. \tag{11.51}$$

In order to simplify the expression for $\breve{\mathcal{F}}$ flux, let us introduce new sets of derivatives by

$$\hat{\partial}_\mu = (\mathcal{C}^T)_\mu^{\ \nu}\partial_\nu\,. \tag{11.52}$$

Here it is worth noting that in general there is no coordinate system $\hat{x}^\mu$ so that $\hat{\partial}_\mu$ are well-defined partial derivatives in that system. This would only be true if the matrix $\mathcal{C}^T$ defined the diffeomorphisms, i.e., if $(\mathcal{C}^T)_\mu^{\ \nu} = \frac{\partial x^\mu}{\partial \hat{x}^\nu}$, which would imply $\hat{\partial}_\mu(\mathcal{C}^T)_\nu^{\ \rho} - \hat{\partial}_\nu(\mathcal{C}^T)_\mu^{\ \rho} = 0$. However, the matrix $\mathcal{C}$ is defined in terms of the string fields B and θ, and this relation does not hold.

Furthermore, we introduce a new non-commutative field $\breve{\theta}$, which is given by

$$\breve{\theta}^{\mu\nu} = (\mathcal{S}\mathcal{C}^{-1})^\mu_{\ \rho}\theta^{\rho\nu} = (C^{-2})^\mu_{\ \rho}\mathring{\theta}^{\rho\nu}\,. \tag{11.53}$$

After substituting (11.41) and (11.53) into (11.51), one obtains

$$\breve{\mathcal{F}}_{\mu\nu}^{\ \rho} = \breve{f}_{\mu\nu}^{\ \rho} - 2\kappa\breve{\mathcal{B}}_{\mu\nu\sigma}\breve{\theta}^{\sigma\rho}\,, \tag{11.54}$$

where

$$\breve{f}_{\mu\nu}^{\ \rho} = (\mathcal{C}^{-1})^\rho_{\ \sigma}\Big(\hat{\partial}_\mu\mathcal{C}^\sigma_{\ \nu} - \hat{\partial}_\nu\mathcal{C}^\sigma_{\ \mu}\Big)\,. \tag{11.55}$$

Similarly, starting with (11.37), with the help of (11.42) and (11.48), we have

$$\begin{aligned}
\{\breve{k}^\mu(\sigma), \breve{k}^\nu(\bar{\sigma})\} &= (\mathcal{C}^{-1})^\mu_{\ \rho}(\mathcal{C}^{-1})^\nu_{\ \sigma}\{\mathring{k}^\rho, \mathring{k}^\sigma\} + (\mathcal{C}^{-1})^\mu_{\ \rho}\mathring{k}^\sigma\{\mathring{k}^\rho, (\mathcal{C}^{-1})^\nu_{\ \sigma}\} \\
&\quad + \mathring{k}^\rho\{(\mathcal{C}^{-1})^\mu_{\ \rho}, \mathring{k}^\sigma\}(\mathcal{C}^{-1})^\nu_{\ \sigma} \\
&= -\kappa(\mathcal{C}^{-1})^\mu_{\ \rho}(\mathcal{C}^{-1})^\nu_{\ \sigma}\mathring{Q}_\tau^{\ \rho\sigma}\mathring{k}^\tau\delta(\sigma-\bar{\sigma}) - \kappa^2(\mathcal{C}^{-1})^\mu_{\ \rho}(\mathcal{C}^{-1})^\nu_{\ \sigma}\mathring{R}^{\rho\sigma\tau}\mathring{\iota}_\tau\delta(\sigma-\bar{\sigma}) \\
&\quad -\kappa\Big(\breve{\theta}^{\mu\alpha}\hat{\partial}_\alpha(\mathcal{C}^{-1})^\nu_{\ \rho} - \breve{\theta}^{\nu\alpha}\hat{\partial}_\alpha(\mathcal{C}^{-1})^\mu_{\ \rho}\Big)\mathring{k}^\rho\delta(\sigma-\bar{\sigma}) \\
&= -\kappa\breve{\mathcal{Q}}_\rho^{\ \mu\nu}\breve{k}^\rho\delta(\sigma-\bar{\sigma}) - \kappa^2\breve{\mathcal{R}}^{\mu\nu\rho}\breve{\iota}_\rho\delta(\sigma-\bar{\sigma})\,,
\end{aligned} \tag{11.56}$$

where we used the relations (11.52) and (11.53) to simplify result. The fluxes that we obtained are

$$\breve{\mathcal{Q}}_\rho^{\ \mu\nu} = \mathcal{C}^\alpha_{\ \rho}(\mathcal{C}^{-1})^\mu_{\ \beta}(\mathcal{C}^{-1})^\nu_{\ \gamma}\mathring{\mathcal{Q}}_\alpha^{\ \beta\gamma} - \mathcal{C}^\alpha_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_\beta(\mathcal{C}^{-1})^\mu_{\ \alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_\beta(\mathcal{C}^{-1})^\nu_{\ \alpha}\Big)\,, \tag{11.57}$$

and

$$\breve{\mathcal{R}}^{\mu\nu\rho} = \mathring{\mathcal{R}}^{\alpha\beta\gamma}(\mathcal{C}^{-1})^{\mu}{}_{\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\gamma}\,. \tag{11.58}$$

These are expressions for analogs of non-geometric fluxes. We will now proceed to rewrite them in more recognizable forms. For $\breve{\mathcal{Q}}$-flux, substituting (11.43) into (11.57), we obtain

$$\breve{\mathcal{Q}}_{\rho}^{\ \mu\nu} = \breve{Q}_{\rho}^{\ \mu\nu} + 2\kappa\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\mathcal{B}}_{\rho\alpha\beta}\,, \tag{11.59}$$

where

$$\breve{Q}_{\rho}^{\ \mu\nu} = \mathcal{C}^{\alpha}_{\ \rho}(\mathcal{C}^{-1})^{\mu}{}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\gamma}\mathring{Q}_{\alpha}^{\ \beta\gamma} - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big)\,. \tag{11.60}$$

After expressing the previous relation in terms of $\breve{\theta}$ (11.53), we obtain

$$\begin{aligned}
\breve{Q}_{\rho}^{\ \mu\nu} &= \partial_{\alpha}(\mathcal{C}^{2}\breve{\theta})^{\beta\gamma}\mathcal{C}^{\alpha}_{\ \rho}(\mathcal{C}^{-1})^{\mu}{}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\gamma} - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big) \qquad (11.61)\\
&= \hat{\partial}_{\rho}\breve{\theta}^{\mu\nu} - \Big((\mathcal{C}\breve{\theta})^{\beta\nu}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\mu}{}_{\beta} + (\mathcal{C}\breve{\theta})^{\mu\gamma}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\nu}{}_{\gamma})\Big) - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big)\\
&= \hat{\partial}_{\rho}\breve{\theta}^{\mu\nu} + \mathcal{C}^{\alpha}_{\ \beta}\breve{\theta}^{\nu\beta}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \mathcal{C}^{\alpha}_{\ \beta}\breve{\theta}^{\mu\beta}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\nu}{}_{\alpha} - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big)\\
&= \hat{\partial}_{\rho}\breve{\theta}^{\mu\nu} + \breve{\theta}^{\nu\beta}\Big(\mathcal{C}^{\alpha}_{\ \beta}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \mathcal{C}^{\alpha}_{\ \rho}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha}\Big) - \breve{\theta}^{\mu\beta}\Big(\mathcal{C}^{\alpha}_{\ \beta}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\nu}{}_{\alpha} - \mathcal{C}^{\alpha}_{\ \rho}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big)\\
&= \hat{\partial}_{\rho}\breve{\theta}^{\mu\nu} + \breve{f}_{\rho\sigma}^{\ \ \mu}\,\breve{\theta}^{\sigma\nu} - \breve{f}_{\rho\sigma}^{\ \ \nu}\,\breve{\theta}^{\sigma\mu}\,.
\end{aligned}$$

In the first step, we expressed the flux $\mathring{Q}$ using the non-commutative field $\breve{\theta}$. Then, in the second step, we used partial integration on the first term and rearranged the resulting expression using equation (11.52). In subsequent steps, we recognized that $(\mathcal{C}\breve{\theta})^{\beta\nu}$ can be expressed as $\mathcal{C}^{\beta}_{\ \sigma}\breve{\theta}^{\sigma\nu}$ and that $\mathcal{C}^{\mu}_{\ \beta}\partial_{\alpha}(\mathcal{C}^{-1})^{\beta}{}_{\sigma}$ equals $-\partial_{\alpha}\mathcal{C}^{\mu}_{\ \beta}(\mathcal{C}^{-1})^{\beta}{}_{\sigma}$. By relabeling some indices and using equation (11.55), we arrived at the final step of equation (11.61).

Similarly, substituting (11.44) into (11.58), we obtain

$$\breve{\mathcal{R}}^{\mu\nu\rho} = \breve{R}^{\mu\nu\rho} + 2\kappa\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\theta}^{\rho\gamma}\breve{\mathcal{B}}_{\alpha\beta\gamma}\,, \tag{11.62}$$

where

$$\breve{R}^{\mu\nu\rho} = \mathring{R}^{\alpha\beta\gamma}(\mathcal{C}^{-1})^{\mu}{}_{\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\gamma} \tag{11.63}$$

The $\breve{R}$-flux can further be rewritten by

$$\begin{aligned}
\breve{R}^{\mu\nu\rho} &= (\mathcal{C}^{2}\breve{\theta})^{\alpha\sigma}\partial_{\sigma}(\mathcal{C}^{2}\breve{\theta})^{\beta\gamma}(\mathcal{C}^{-1})^{\mu}{}_{\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\gamma} + cyclic \qquad (11.64)\\
&= (\mathcal{C}\breve{\theta})^{\mu\sigma}\partial_{\sigma}\breve{\theta}^{\nu\rho} - (\mathcal{C}\breve{\theta})^{\mu\sigma}(\mathcal{C}\breve{\theta})^{\beta\gamma}(\partial_{\sigma}(\mathcal{C}^{-1})^{\nu}{}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\gamma} + (\mathcal{C}^{-1})^{\nu}{}_{\beta}\partial_{\sigma}(\mathcal{C}^{-1})^{\rho}{}_{\gamma}) + cyclic\\
&= \breve{\theta}^{\mu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\nu\rho} + \breve{\theta}^{\mu\alpha}\breve{\theta}^{\rho\beta}\hat{\partial}_{\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\tau}\mathcal{C}^{\tau}_{\ \beta} - \breve{\theta}^{\mu\beta}\breve{\theta}^{\nu\alpha}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\tau}\mathcal{C}^{\tau}_{\ \alpha} + cyclic\\
&= \breve{\theta}^{\mu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\nu\rho} + \breve{\theta}^{\nu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\rho\mu} + \breve{\theta}^{\rho\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\mu\nu} - \breve{\theta}^{\mu\alpha}\breve{\theta}^{\rho\beta}(\mathcal{C}^{-1})^{\nu}{}_{\tau}\hat{\partial}_{\alpha}\mathcal{C}^{\tau}_{\ \beta} + \breve{\theta}^{\mu\beta}\breve{\theta}^{\nu\alpha}(\mathcal{C}^{-1})^{\rho}{}_{\tau}\hat{\partial}_{\beta}\mathcal{C}^{\tau}_{\ \alpha}\\
&\quad -\breve{\theta}^{\nu\alpha}\breve{\theta}^{\mu\beta}(\mathcal{C}^{-1})^{\rho}{}_{\tau}\hat{\partial}_{\alpha}\mathcal{C}^{\tau}_{\ \beta} + \breve{\theta}^{\nu\beta}\breve{\theta}^{\rho\alpha}(\mathcal{C}^{-1})^{\mu}{}_{\tau}\hat{\partial}_{\beta}\mathcal{C}^{\tau}_{\ \alpha} - \breve{\theta}^{\rho\alpha}\breve{\theta}^{\nu\beta}(\mathcal{C}^{-1})^{\mu}{}_{\tau}\hat{\partial}_{\alpha}\mathcal{C}^{\tau}_{\ \beta} + \breve{\theta}^{\rho\beta}\breve{\theta}^{\mu\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\tau}\hat{\partial}_{\beta}\mathcal{C}^{\tau}_{\ \alpha}\\
&= \breve{\theta}^{\mu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\nu\rho} + \breve{\theta}^{\nu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\rho\mu} + \breve{\theta}^{\rho\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\mu\nu} - (\breve{\theta}^{\mu\alpha}\breve{\theta}^{\rho\beta}\breve{f}_{\alpha\beta}^{\ \ \nu} + \breve{\theta}^{\nu\alpha}\breve{\theta}^{\mu\beta}\breve{f}_{\alpha\beta}^{\ \ \rho} + \breve{\theta}^{\rho\alpha}\breve{\theta}^{\nu\beta}\breve{f}_{\alpha\beta}^{\ \ \mu})\,.
\end{aligned}$$

First, we expressed the flux as a function of $\breve{\theta}$ according to equation (11.53). Then, we applied the chain rule and the derivative $\hat{\partial}$ (11.52). Finally, we applied the chain rule again to hyperbolic functions and used equation (11.55) to obtain the final expression.

Lastly, the remaining algebra between currents is obtained from (11.45) and (11.48) in a following way

$$\begin{aligned}\{\breve{\imath}_\mu(\sigma),\breve{k}^\nu(\bar{\sigma})\} &= \mathcal{C}^\sigma{}_\mu(\mathcal{C}^{-1})^\nu{}_\rho\{\mathring{\imath}_\sigma,\mathring{k}^\rho(\bar{\sigma})\}+\mathcal{C}^\sigma{}_\mu\{\mathring{\imath}_\sigma,(\mathcal{C}^{-1})^\nu{}_\rho\}\mathring{k}^\rho+\mathring{\imath}_\sigma\{\mathcal{C}^\sigma{}_\mu,\mathring{k}^\rho\}(\mathcal{C}^{-1})^\nu{}_\rho \\ &= \kappa\mathcal{C}^\sigma{}_\mu(\sigma)(\mathcal{C}^{-1})^\nu{}_\sigma(\bar{\sigma})\delta'(\sigma-\bar{\sigma})+\Big(\mathring{\mathcal{F}}_{\rho\tau}{}^\sigma\,\mathcal{C}^\rho{}_\mu(\mathcal{C}^{-1})^\nu{}_\sigma-\hat{\partial}_\mu(\mathcal{C}^{-1})^\nu{}_\tau\Big)\mathring{k}^\tau\delta(\sigma-\bar{\sigma}) \\ &\quad+\Big(-\kappa\mathring{\mathcal{Q}}_\rho{}^{\sigma\tau}\mathcal{C}^\rho{}_\mu(\mathcal{C}^{-1})^\nu{}_\sigma+\kappa(\mathcal{C}^{-1})^\nu{}_\sigma\mathring{\theta}^{\sigma\rho}\partial_\rho\mathcal{C}^\tau{}_\mu\Big)\mathring{\imath}_\tau\delta(\sigma-\bar{\sigma})\,.\end{aligned} \tag{11.65}$$

Using (7.7), the anomalous term becomes

$$\begin{aligned}\kappa\mathcal{C}^\sigma{}_\mu(\sigma)(\mathcal{C}^{-1})^\nu{}_\sigma(\bar{\sigma})\delta'(\sigma-\bar{\sigma}) &= \kappa\delta^\nu_\mu\delta'(\sigma-\bar{\sigma})+\kappa\mathcal{C}^\sigma{}_\mu\partial_\rho(\mathcal{C}^{-1})^\nu{}_\sigma x'^\rho\delta(\sigma-\bar{\sigma}) \\ &= \kappa\delta^\nu_\mu\delta'(\sigma-\bar{\sigma})+\mathcal{C}^\rho{}_\mu\partial_\sigma(\mathcal{C}^{-1})^\nu{}_\rho\mathring{k}^\sigma\delta(\sigma-\bar{\sigma}) \\ &\quad-\kappa\mathcal{C}^\rho{}_\mu\partial_\sigma(\mathcal{C}^{-1})^\nu{}_\rho\mathring{\theta}^{\sigma\tau}\mathring{\imath}_\tau\delta(\sigma-\bar{\sigma})\,,\end{aligned} \tag{11.66}$$

where we also used (11.38). Substituting (11.66) into (11.65), we obtain

$$\begin{aligned}\{\breve{\imath}_\mu(\sigma),\breve{k}^\nu(\bar{\sigma})\} &= \kappa\delta^\nu_\mu\delta'(\sigma-\bar{\sigma}) \\ &\quad+\Big(\mathcal{C}^\rho{}_\mu(\partial_\sigma(\mathcal{C}^{-1})^\nu{}_\rho-\partial_\rho(\mathcal{C}^{-1})^\nu{}_\sigma)+\mathcal{C}^\rho{}_\mu(\mathcal{C}^{-1})^\nu{}_\tau\mathring{\mathcal{F}}_{\rho\sigma}{}^\tau\Big)\mathring{k}^\sigma\delta(\sigma-\bar{\sigma}) \\ &\quad+\kappa\Big((\mathcal{C}^{-1})^\nu{}_\sigma\partial_\rho\mathcal{C}^\tau{}_\mu\mathring{\theta}^{\sigma\rho}-\mathcal{C}^\rho{}_\mu\partial_\sigma(\mathcal{C}^{-1})^\nu{}_\rho\mathring{\theta}^{\sigma\tau}-\mathcal{C}^\rho{}_\mu(\mathcal{C}^{-1})^\nu{}_\sigma\mathring{\mathcal{Q}}_\rho{}^{\sigma\tau}\Big)\mathring{\imath}_\tau\delta(\sigma-\bar{\sigma})\,.\end{aligned} \tag{11.67}$$

Substituting relations between currents in the previous expression (11.37), we obtain

$$\{\breve{\imath}_\mu(\sigma),\breve{k}^\nu(\bar{\sigma})\}=\kappa\delta^\nu_\mu\delta'(\sigma-\bar{\sigma})+\breve{\mathcal{F}}_{\mu\rho}{}^\nu\,\breve{k}^\rho\delta(\sigma-\bar{\sigma})-\kappa\breve{\mathcal{Q}}_\mu{}^{\nu\rho}\breve{\imath}_\rho\delta(\sigma-\bar{\sigma})\,, \tag{11.68}$$

All Poisson bracket relations between currents (11.49), (11.56), (11.68) can be summarized by

$$\{\breve{X}^M,\breve{X}^N\}=-\breve{F}^{MN}{}_P\,\breve{X}^P\delta(\sigma-\bar{\sigma})+\kappa\eta^{MN}\delta'(\sigma-\bar{\sigma})\,, \tag{11.69}$$

where

$$\breve{F}^{MN\rho}=\begin{pmatrix}\kappa^2\breve{\mathcal{R}}^{\mu\nu\rho} & -\kappa\breve{\mathcal{Q}}_\nu{}^{\mu\rho} \\ \kappa\breve{\mathcal{Q}}_\mu{}^{\nu\rho} & \breve{\mathcal{F}}_{\mu\nu}{}^\rho\end{pmatrix}\,,\qquad \breve{F}^{MN}{}_\rho=\begin{pmatrix}\kappa\breve{\mathcal{Q}}_\rho{}^{\mu\nu} & \breve{\mathcal{F}}_{\nu\rho}{}^\mu \\ -\breve{\mathcal{F}}_{\mu\rho}{}^\nu & 2\breve{\mathcal{B}}_{\mu\nu\rho}\end{pmatrix}\,. \tag{11.70}$$

With basic algebra relations, we can now proceed with calculations of the Courant bracket twisted simultaneously by B and θ.

11.5 Full bracket

The full bracket can be obtained from Poisson bracket relations of the double generator (11.30)

$$\{\breve{\mathcal{G}}_{\Lambda_1}, \breve{\mathcal{G}}_{\Lambda_2}\} = -\breve{\mathcal{G}}_{[\Lambda_1,\Lambda_2]_{\mathcal{C}_{\breve{B}}}} . \tag{11.71}$$

We rewrite term-wise the left hand side of the previous relation

$$\begin{aligned}\{\breve{\mathcal{G}}_{\Lambda_1}, \breve{\mathcal{G}}_{\Lambda_2}\} &= \int d\sigma d\bar{\sigma}\Big(\{\xi_1^\mu(\sigma)\breve{\iota}_\mu(\sigma), \xi_2^\nu(\bar{\sigma})\breve{\iota}_\nu(\bar{\sigma})\} + \{\lambda_{1\mu}(\sigma)\breve{k}^\mu(\sigma), \lambda_{2\nu}(\bar{\sigma})\breve{k}^\nu(\bar{\sigma})\} \\ &\quad +\{\xi_1^\mu(\sigma)\breve{\iota}_\mu(\sigma), \lambda_{2\nu}(\bar{\sigma})\breve{k}^\nu(\bar{\sigma})\} + \{\lambda_{1\mu}(\sigma)\breve{k}^\mu(\sigma), \xi_2^\nu(\bar{\sigma})\breve{\iota}_\nu(\bar{\sigma})\}\Big) .\end{aligned} \tag{11.72}$$

Apart from the relations between currents, other Poisson bracket relations that we will use are

$$\{\xi_1^\mu(\sigma), \breve{\iota}_\nu(\bar{\sigma})\} = (\mathcal{C}^T)_\nu{}^\rho\partial_\rho\xi_1^\mu\delta(\sigma-\bar{\sigma}) = \hat{\partial}_\nu\xi_1^\mu\delta(\sigma-\bar{\sigma}) , \tag{11.73}$$

$$\{\xi_1^\mu(\sigma), \breve{k}^\nu(\bar{\sigma})\} = \kappa(\mathcal{C}\breve{\theta})^{\nu\rho}\partial_\rho\xi_1^\mu\delta(\sigma-\bar{\sigma}) = \kappa\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\xi_1^\mu\delta(\sigma-\bar{\sigma}) . \tag{11.74}$$

Using (11.49) and (11.73), the first term of (11.72) becomes

$$\begin{aligned}\{\xi_1^\mu(\sigma)\breve{\iota}_\mu(\sigma), \xi_2^\nu(\bar{\sigma})\breve{\iota}_\nu(\bar{\sigma})\} &= (\xi_2^\nu\hat{\partial}_\nu\xi_1^\mu - \xi_1^\nu\hat{\partial}_\nu\xi_2^\mu - \breve{\mathcal{F}}_{\nu\rho}{}^\mu\,\xi_1^\nu\xi_2^\rho)\breve{\iota}_\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -2\breve{\mathcal{B}}_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho\breve{k}^\mu\delta(\sigma-\bar{\sigma}) ,\end{aligned} \tag{11.75}$$

which after we substitute (11.54) becomes

$$\begin{aligned}\{\xi_1^\mu(\sigma)\breve{\iota}_\mu(\sigma), \xi_2^\nu(\bar{\sigma})\breve{\iota}_\nu(\bar{\sigma})\} &= (\xi_2^\nu\hat{\partial}_\nu\xi_1^\mu - \xi_1^\nu\hat{\partial}_\nu\xi_2^\mu - (\breve{f}_{\nu\rho}{}^\mu - 2\kappa\breve{\mathcal{B}}_{\nu\rho\sigma}\breve{\theta}^{\sigma\mu})\xi_1^\nu\xi_2^\rho)\breve{\iota}_\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -2\breve{\mathcal{B}}_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho\breve{k}^\mu\delta(\sigma-\bar{\sigma}) ,\end{aligned} \tag{11.76}$$

and similarly, from (11.56) and (11.74), we obtain

$$\begin{aligned}\{\lambda_{1\mu}(\sigma)\breve{k}^\mu(\sigma), \lambda_{2\nu}(\bar{\sigma})\breve{k}^\nu(\bar{\sigma})\} &= -\kappa^2\breve{\mathcal{R}}^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\breve{\iota}_\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -\kappa(\breve{\mathcal{Q}}_\mu{}^{\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + \kappa\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\lambda_{2\mu}\lambda_{1\nu} - \kappa\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\lambda_{1\mu}\lambda_{2\nu})\breve{k}^\mu\delta(\sigma-\bar{\sigma}) .\end{aligned} \tag{11.77}$$

which after substituting (11.59) and (11.62) becomes

$$\begin{aligned}\{\lambda_{1\mu}(\sigma)\breve{k}^\mu(\sigma), \lambda_{2\nu}(\bar{\sigma})\breve{k}^\nu(\bar{\sigma})\} &= -\kappa^2\Big(\breve{R}^{\mu\nu\rho} + 2\kappa\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\theta}^{\rho\gamma}\breve{\mathcal{B}}_{\alpha\beta\gamma}\Big)\lambda_{1\nu}\lambda_{2\rho}\breve{\iota}_\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -\kappa\Big(\hat{\partial}_\mu\breve{\theta}^{\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + \breve{\theta}^{\nu\rho}\hat{\partial}_\rho\lambda_{2\mu}\lambda_{1\nu} - \breve{\theta}^{\nu\rho}\hat{\partial}_\rho\lambda_{1\mu}\lambda_{2\nu} \\ &\quad +(\breve{f}_{\mu\sigma}{}^\nu\,\breve{\theta}^{\sigma\rho} - \breve{f}_{\mu\sigma}{}^\rho\,\breve{\theta}^{\sigma\nu} + 2\kappa\breve{\theta}^{\nu\alpha}\breve{\theta}^{\rho\beta}\breve{\mathcal{B}}_{\mu\alpha\beta})\lambda_{1\nu}\lambda_{2\rho}\Big)\breve{k}^\mu\delta(\sigma-\bar{\sigma}) .\end{aligned} \tag{11.78}$$

Lastly, using (11.68), (11.73), and (11.74), one obtains

$$\begin{aligned}\{\xi_1^\mu(\sigma)\breve{\iota}_\mu(\sigma),\lambda_{2\nu}(\bar\sigma)\breve{k}^\nu(\bar\sigma)\} &= \kappa\xi_1^\mu(\sigma)\lambda_{2\mu}(\bar\sigma)\delta'(\sigma-\bar\sigma)\\ &+(\breve{\mathcal{F}}_{\nu\mu}{}^{\rho}\,\xi_1^\nu\lambda_{2\rho}-\xi_1^\nu\hat\partial_\nu\lambda_{2\mu})\breve{k}^\mu\delta(\sigma-\bar\sigma)\\ &+(-\kappa\breve{\mathcal{Q}}_\rho{}^{\nu\mu}\xi_1^\rho\lambda_{2\nu}+\kappa\lambda_{2\nu}\breve\theta^{\nu\rho}\hat\partial_\rho\xi_1^\mu)\breve{\iota}_\mu\delta(\sigma-\bar\sigma)\,.\end{aligned}\tag{11.79}$$

The anomalous part depends on both σ and $\bar\sigma$, so it should be further modified by

$$\begin{aligned}\kappa\xi_1^\mu(\sigma)\lambda_{2\mu}(\bar\sigma)\delta'(\sigma-\bar\sigma) &= \frac{\kappa}{2}\xi_1^\mu(\sigma)\lambda_{2\mu}(\bar\sigma)\delta'(\sigma-\bar\sigma)-\frac{\kappa}{2}\xi_1^\mu(\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)\\ &= \frac{\kappa}{2}\xi_1^\mu\lambda_{2\mu}\delta'(\sigma-\bar\sigma)+\frac{\kappa}{2}\xi_1^\mu\partial_\nu\lambda_{2\mu}x'^\nu\delta(\sigma-\bar\sigma)\\ &-\frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)-\frac{\kappa}{2}\partial_\nu\xi_1^\mu\lambda_{2\mu}x'^\nu\delta(\sigma-\bar\sigma)\\ &= \frac{\kappa}{2}\xi_1^\mu\lambda_{2\mu}\delta'(\sigma-\bar\sigma)-\frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)\\ &+\frac{\kappa}{2}(\xi_1^\rho\hat\partial_\nu\lambda_{2\rho}\breve\theta^{\nu\mu}-\hat\partial_\nu\xi_1^\rho\lambda_{2\rho}\breve\theta^{\nu\mu})\breve\iota_\mu\delta(\sigma-\bar\sigma)\\ &+\frac{1}{2}(\xi_1^\nu\hat\partial_\mu\lambda_{2\nu}-\hat\partial_\mu\xi_1^\nu\lambda_{2\nu})\breve{k}^\mu\delta(\sigma-\bar\sigma)\\ &= \frac{\kappa}{2}\xi_1^\mu\lambda_{2\mu}\delta'(\sigma-\bar\sigma)-\frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)\\ &+\kappa\breve\theta^{\mu\nu}(\xi_1^\rho\hat\partial_\nu\lambda_{2\rho}-\frac{1}{2}\hat\partial_\nu(\xi_1^\rho\lambda_{2\rho}))\breve\iota_\mu\delta(\sigma-\bar\sigma)\\ &+(\xi_1^\nu\hat\partial_\mu\lambda_{2\nu}-\frac{1}{2}\hat\partial_\mu(\xi_1^\nu\lambda_{2\nu}))\breve{k}^\mu\delta(\sigma-\bar\sigma)\,.\end{aligned}\tag{11.80}$$

We used the property of the delta function (7.7) in the initial two steps, followed by using the relation (11.29) in the subsequent step, and eventually, we applied the chain rule in the final step. After substituting (11.80) into (11.79), we obtain

$$\begin{aligned}\{\xi_1^\nu(\sigma)\breve\iota_\nu(\sigma),\lambda_{2\mu}(\bar\sigma)\breve{k}^\mu(\bar\sigma)\} &= \frac{\kappa}{2}\xi_1^\mu\lambda_{2\mu}\delta'(\sigma-\bar\sigma)-\frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)\\ &+\Big(\breve{\mathcal{F}}_{\nu\mu}{}^{\rho}\,\xi_1^\nu\lambda_{2\rho}+\xi_1^\nu(\hat\partial_\mu\lambda_{2\nu}-\hat\partial_\nu\lambda_{2\mu})-\frac{1}{2}\hat\partial_\mu(\xi_1^\nu\lambda_{2\nu})\Big)\breve{k}^\mu\delta(\sigma-\bar\sigma)\\ &+\Big(-\kappa\breve{\mathcal{Q}}_\rho{}^{\nu\mu}\xi_1^\rho\lambda_{2\nu}+\kappa\lambda_{2\nu}\breve\theta^{\nu\rho}\hat\partial_\rho\xi_1^\mu\\ &\quad+\kappa\breve\theta^{\mu\nu}(\xi_1^\rho\hat\partial_\nu\lambda_{2\rho}-\frac{1}{2}\hat\partial_\nu(\xi_1^\rho\lambda_{2\rho}))\Big)\breve\iota_\mu\delta(\sigma-\bar\sigma)\,.\end{aligned}\tag{11.81}$$

To obtain more recognizable terms, we will use the chain rule in order to transform the term containing $\breve{\mathcal{Q}}$ flux (11.59)

$$\begin{aligned}-\kappa\breve{\mathcal{Q}}_\rho{}^{\nu\mu}\xi_1^\rho\lambda_{2\nu} &= -\kappa\Big(\hat\partial_\rho\breve\theta^{\nu\mu}+\breve{f}_{\rho\sigma}{}^{\nu}\breve\theta^{\sigma\mu}-\breve{f}_{\rho\sigma}{}^{\mu}\breve\theta^{\sigma\nu}+2\kappa^2\breve\theta^{\nu\alpha}\breve\theta^{\mu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}\Big)\xi_1^\rho\lambda_{2\nu}\\ &= -\kappa\xi_1^\rho\hat\partial_\rho(\lambda_{2\nu}\breve\theta^{\nu\mu})+\kappa\xi_1^\rho\hat\partial_\rho\lambda_{2\nu}\breve\theta^{\nu\mu}\\ &\quad-\kappa\Big(\breve{f}_{\rho\sigma}{}^{\nu}\breve\theta^{\sigma\mu}-\breve{f}_{\rho\sigma}{}^{\mu}\breve\theta^{\sigma\nu}+2\kappa^2\breve\theta^{\nu\alpha}\breve\theta^{\mu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}\Big)\xi_1^\rho\lambda_{2\nu}\,.\end{aligned}\tag{11.82}$$

Substituting (11.54) and (11.82) into (11.81), we obtain

$$\begin{aligned}
\{\xi_1^\nu \breve{\iota}_\nu, \lambda_{2\mu}(\bar{\sigma})\breve{k}^\mu(\bar{\sigma})\} &= \frac{\kappa}{2}\xi_1^\mu \lambda_{2\mu}\delta'(\sigma-\bar{\sigma}) - \frac{\kappa}{2}\xi_1^\mu(\bar{\sigma})\lambda_{2\mu}(\bar{\sigma})\partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma}) \\
&+\Big(\xi_1^\nu(\hat{\partial}_\mu\lambda_{2\nu} - \hat{\partial}_\nu\lambda_{2\mu}) - \frac{1}{2}\hat{\partial}_\mu(\xi_1^\nu\lambda_{2\nu}) \\
&+(\breve{f}_{\nu\mu}^{\ \ \rho} - 2\kappa\breve{\mathcal{B}}_{\nu\mu\sigma}\breve{\theta}^{\sigma\rho})\xi_1^\nu\lambda_{2\rho}\Big)\breve{k}^\mu\delta(\sigma-\bar{\sigma}) \\
&+\Big[\kappa\breve{\theta}^{\mu\nu}\Big(\xi_1^\rho(\hat{\partial}_\nu\lambda_{2\rho} - \hat{\partial}_\rho\lambda_{2\nu}) - \frac{1}{2}\hat{\partial}_\nu(\xi_1^\rho\lambda_{2\rho})\Big) \\
&+\kappa\lambda_{2\nu}\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\xi_1^\mu - \kappa\xi_1^\rho\hat{\partial}_\rho(\lambda_{2\nu}\breve{\theta}^{\nu\mu}) \\
&-\kappa\Big(\breve{f}_{\rho\sigma}^{\ \ \nu}\breve{\theta}^{\sigma\mu} - \breve{f}_{\rho\sigma}^{\ \ \mu}\breve{\theta}^{\sigma\nu} + 2\kappa^2\breve{\theta}^{\nu\alpha}\breve{\theta}^{\mu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}\Big)\xi_1^\rho\lambda_{2\nu}\Big]\breve{\iota}_\mu\delta(\sigma-\bar{\sigma}) \,.
\end{aligned} \tag{11.83}$$

Substituting (11.76), (11.78), and (11.83) into (11.72), with the help of (11.71), we obtain

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_{\breve{B}}} = \Lambda = \xi \oplus \lambda \,, \tag{11.84}$$

where

$$\begin{aligned}
\xi^\mu &= \xi_1^\nu\hat{\partial}_\nu\xi_2^\mu - \xi_2^\nu\hat{\partial}_\nu\xi_1^\mu + \breve{f}_{\nu\rho}^{\ \ \mu}\,\xi_1^\nu\xi_2^\rho \\
&+\kappa\breve{\theta}^{\mu\nu}\Big(\xi_1^\rho(\hat{\partial}_\rho\lambda_{2\nu} - \hat{\partial}_\nu\lambda_{2\rho}) - \xi_2^\rho(\hat{\partial}_\rho\lambda_{1\nu} - \hat{\partial}_\nu\lambda_{1\rho}) + \frac{1}{2}\hat{\partial}_\nu(\xi_1\lambda_2 - \xi_2\lambda_1) \\
&\qquad +\kappa\breve{f}_{\nu\rho}^{\ \ \sigma}(\xi_1^\rho\lambda_{2\nu} - \xi_2^\rho\lambda_{1\nu})\Big) \\
&+\kappa\xi_1^\nu\hat{\partial}_\nu(\lambda_{2\rho}\breve{\theta}^{\rho\mu}) - \kappa\xi_2^\nu\hat{\partial}_\nu(\lambda_{1\rho}\breve{\theta}^{\rho\mu}) - \kappa\lambda_{2\nu}\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\xi_1^\mu + \kappa\lambda_{1\nu}\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\xi_2^\mu \\
&\qquad +\kappa\breve{f}_{\rho\sigma}^{\ \ \nu}\,\breve{\theta}^{\sigma\mu}(\xi_1^\rho\lambda_{2\nu} - \xi_2^\rho\lambda_{1\nu}) \\
&+\kappa^2\breve{R}^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho} \\
&-2\kappa\breve{B}_{\nu\rho\sigma}\breve{\theta}^{\sigma\mu}\xi_1^\nu\xi_2^\rho + 2\kappa^2\breve{\theta}^{\nu\alpha}\breve{\theta}^{\mu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}(\xi_1^\rho\lambda_{2\nu} - \xi_2^\rho\lambda_{1\nu}) + 2\kappa^3\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\theta}^{\rho\gamma}\breve{\mathcal{B}}_{\alpha\beta\gamma}\lambda_{1\nu}\lambda_{2\rho} \,,
\end{aligned} \tag{11.85}$$

and

$$\begin{aligned}
\lambda_\mu &= \xi_1^\nu(\hat{\partial}_\nu\lambda_{2\mu} - \hat{\partial}_\mu\lambda_{2\nu}) - \xi_2^\nu(\hat{\partial}_\nu\lambda_{1\mu} - \hat{\partial}_\mu\lambda_{1\nu}) + \frac{1}{2}\hat{\partial}_\mu(\xi_1\lambda_2 - \xi_2\lambda_1) \\
&+\breve{f}_{\mu\nu}^{\ \ \rho}(\xi_1^\nu\lambda_{2\rho} - \xi_2^\nu\lambda_{1\rho}) \\
&+\kappa\breve{\theta}^{\nu\rho}(\lambda_{1\nu}\hat{\partial}_\rho\lambda_{2\mu} - \lambda_{2\nu}\hat{\partial}_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}\hat{\partial}_\mu\breve{\theta}^{\rho\nu} + \kappa(\breve{f}_{\mu\sigma}^{\ \ \nu}\,\breve{\theta}^{\sigma\rho} - \breve{f}_{\mu\sigma}^{\ \ \rho}\,\breve{\theta}^{\sigma\nu})\lambda_{1\nu}\lambda_{2\rho} \\
&+2\breve{\mathcal{B}}_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho - 2\kappa\breve{\mathcal{B}}_{\mu\nu\sigma}\breve{\theta}^{\sigma\rho}(\xi_1^\nu\lambda_{2\rho} - \xi_2^\nu\lambda_{1\rho}) + 2\kappa^2\breve{\theta}^{\nu\alpha}\breve{\theta}^{\rho\beta}\breve{\mathcal{B}}_{\mu\alpha\beta}\lambda_{1\nu}\lambda_{2\rho} \,.
\end{aligned} \tag{11.86}$$

We grouped the terms in expressions (11.85) and (11.86) for future convenience.

In the process of twisting the Courant bracket simultaneously by B and θ, we did not rely on the fact that these fields are T-dual background fields. Thus, the results should be valid regardless of this property. If we take that either of the fields B and θ is zero, the derivative $\hat{\partial}_\mu$ reduces to the ordinary partial derivative ∂_μ, and all the $\breve{f}$ flux terms become zero. Specifically, for $\theta = 0$ and an arbitrary B, we obtain B-twisted Courant bracket, while for $B = 0$ and an arbitrary θ, we obtain the θ-twisted Courant bracket.

11.6 Coordinate-free notation

In contrast to prior instances, obtaining the coordinate-free notation with a clear interpretation of its terms is not as trivial for this bracket. We will need to introduce novel brackets that will ultimately be identified as the brackets of Lie and quasi-Lie algebroids that have not been encountered previously.

11.6.1 Twisted Lie bracket

Firstly, we will seek the Lie algebroid with $\mathcal{C}$ as its anchor. This step will turn out to be crucial for interpreting many terms that appear in the expressions (11.85) and (11.86). The bracket of this Lie algebroid should be related to the Lie bracket by

$$\begin{aligned} \Big(\mathcal{C}[\xi_1, \xi_2]_{\hat{L}}\Big)^\mu &= \Big([\mathcal{C}\xi_1, \mathcal{C}\xi_2]_L\Big)^\mu = \mathcal{C}^\nu{}_\rho \xi_1^\rho \partial_\nu (\mathcal{C}^\mu{}_\sigma \xi_2^\sigma) - \mathcal{C}^\nu{}_\rho \xi_2^\rho \partial_\nu (\mathcal{C}^\mu{}_\sigma \xi_1^\sigma) \\ &= \mathcal{C}^\nu{}_\rho \mathcal{C}^\mu{}_\sigma \Big(\xi_1^\rho \partial_\nu \xi_2^\sigma - \xi_2^\rho \partial_\nu \xi_1^\sigma\Big) + \xi_1^\rho \xi_2^\sigma \Big(\mathcal{C}^\nu{}_\rho \partial_\nu \mathcal{C}^\mu{}_\sigma - \mathcal{C}^\nu{}_\sigma \partial_\nu \mathcal{C}^\mu{}_\rho\Big) \\ &= \mathcal{C}^\mu{}_\sigma \Big(\xi_1^\rho \hat{\partial}_\rho \xi_2^\sigma - \xi_2^\rho \hat{\partial}_\rho \xi_1^\sigma\Big) + \xi_1^\rho \xi_2^\sigma \Big(\hat{\partial}_\rho \mathcal{C}^\mu{}_\sigma - \hat{\partial}_\sigma \mathcal{C}^\mu{}_\rho\Big) \,, \end{aligned} \tag{11.87}$$

where we used (11.52) and relabeled some indices. Multiplying the previous relation with $\mathcal{C}^{-1}$ and taking into the account (11.55), we obtain

$$\Big([\xi_1, \xi_2]_{\hat{L}}\Big)^\mu = \xi_1^\nu \hat{\partial}_\nu \xi_2^\mu - \xi_2^\nu \hat{\partial}_\nu \xi_1^\mu + \breve{f}_{\nu\rho}{}^\mu \, \xi_1^\nu \xi_2^\rho \,, \tag{11.88}$$

which is exactly the first line of (11.85). Analogous to our notation for twisted Courant brackets, we will denote this bracket as the twisted Lie bracket, since

$$[\xi_1, \xi_2]_{\hat{L}} = \mathcal{C}^{-1}[\mathcal{C}\xi_1, \mathcal{C}\xi_2]_L \,. \tag{11.89}$$

In order for $\mathcal{C}$ to be a proper anchor of a Lie algebroid, the Leibniz rule has to be satisfied, i.e.

$$[\xi_1, f\xi_2]_{\hat{L}} = (\mathcal{L}_{\mathcal{C}\xi_1} f)\, \xi_2 + f[\xi_1, \xi_2]_{\hat{L}} \,, \tag{11.90}$$

from which we can derive the action of corresponding Lie derivative on functions

$$\hat{\mathcal{L}}_\xi \, f = \mathcal{L}_{\mathcal{C}\xi} \, f = \xi^\mu \hat{\partial}_\mu f \,. \tag{11.91}$$

Its action on vectors is simply given by the twisted Lie bracket. The Jacobi identity is also satisfied, since

$$\text{Jac}_{\hat{L}}(\xi_1, \xi_2, \xi_3) = \mathcal{C}^{-1}[\mathcal{C}\xi_1, [\mathcal{C}\xi_2, \mathcal{C}\xi_3]_L]_L + cyclic = \mathcal{C}^{-1}\text{Jac}_L(\mathcal{C}\xi_1, \mathcal{C}\xi_2, \mathcal{C}\xi_3) = 0\,. \tag{11.92}$$

To write the action of Lie derivative $\mathcal{L}_{\hat{\xi}}$ on 1-forms, we firstly apply the Leibniz rule (11.90) on 1-form-vector contraction

$$\begin{aligned} \hat{\mathcal{L}}_{\xi_1}(\xi_2^\mu \lambda_{2\mu}) &= (\hat{\mathcal{L}}_{\xi_1}\xi_2)^\mu \lambda_{2\mu} + \xi_2^\mu (\hat{\mathcal{L}}_{\xi_1}\lambda_2)_\mu \\ &= (\xi_1^\nu \hat{\partial}_\nu \xi_2^\mu - \xi_2^\nu \hat{\partial}_\nu \xi_1^\mu)\lambda_{2\mu} + \breve{f}_{\nu\rho}{}^\mu\, \xi_1^\nu \xi_2^\rho \lambda_{2\mu} + \xi_2^\mu (\hat{\mathcal{L}}_{\xi_1}\lambda_2)_\mu\,, \end{aligned} \tag{11.93}$$

and then (11.91) on that contraction, since it is effectively a scalar

$$\hat{\mathcal{L}}_{\xi_1}(\xi_2^\mu \lambda_{2\mu}) = \xi_1^\nu \hat{\partial}_\nu (\xi_2^\mu \lambda_{2\mu}) = \xi_1^\nu \hat{\partial}_\nu \xi_2^\mu \lambda_{2\mu} + \xi_1^\nu \xi_2^\mu \hat{\partial}_\nu \lambda_{2\mu}\,. \tag{11.94}$$

When we equate right-hand sides of previous two relations, we obtain

$$(\hat{\mathcal{L}}_{\xi_1}\lambda_2)_\mu = \hat{\partial}_\mu \xi_1^\nu \lambda_{2\nu} + \xi_1^\nu \hat{\partial}_\nu \lambda_{2\mu} + \breve{f}_{\mu\nu}{}^\rho\, \xi_1^\nu \lambda_{2\rho} = \xi_1^\nu(\hat{\partial}_\nu \lambda_{2\mu} - \hat{\partial}_\mu \lambda_{2\nu}) + \hat{\partial}_\mu(\xi_1^\nu \lambda_{2\nu}) + \breve{f}_{\mu\nu}{}^\rho\, \xi_1^\nu \lambda_{2\rho}\,. \tag{11.95}$$

The exterior algebra is easily derived from the relation (5.7). Let us explicitly obtain the action of exterior derivative on functions and 1-forms. Functions correspond to the case $p = 0$, so we have

$$\hat{d}f(\xi) = \mathcal{C}\xi(f) = \xi^\mu \hat{\partial}_\mu f\,, \quad (\hat{d}f)_\mu = \hat{\partial}_\mu f\,. \tag{11.96}$$

From (11.91), we see that the usual relation for the action of Lie derivatives on function $\hat{\mathcal{L}}_\xi f = i_\xi \hat{d} f$ still holds in the twisted case. On the other hand, 1-forms correspond to the case of $p = 1$ in (5.7), from which we obtain

$$\begin{aligned} \hat{d}\lambda(\xi_1, \xi_2) &= \mathcal{C}\xi_1(\lambda(\xi_2)) - \mathcal{C}\xi_2(\lambda(\xi_1)) - \lambda([\xi_1, \xi_2]_{\hat{L}}) \\ &= \xi_1^\mu \xi_2^\nu \Big(\hat{\partial}_\mu \lambda_\nu - \hat{\partial}_\nu \lambda_\mu - \hat{f}_{\mu\nu}{}^\rho \lambda_\rho\Big)\,, \\ (\hat{d}\lambda)_{\mu\nu} &= \hat{\partial}_\mu \lambda_\nu - \hat{\partial}_\nu \lambda_\mu - \hat{f}_{\mu\nu}{}^\rho \lambda_\rho\,. \end{aligned} \tag{11.97}$$

The Cartan formula $\hat{\mathcal{L}}_\xi \lambda = i_\xi \hat{d}\lambda + \hat{d} i_\xi \lambda$ can be easily demonstrated using (11.95) and (11.97), and holds true for any p-form λ.

We saw how the hyperbolic function $\mathcal{C}$ defines an anchor for the Lie algebroid defined with the twisted Lie bracket as its bracket. Various terms in the expressions (11.85) and (11.86) can be expressed in terms of the corresponding twisted Lie derivative.

11.6.2 Generalized H-flux

Let us explore the generalized H-flux $\breve{\mathcal{B}}$ (11.50). It has a structure of a 3-form, that when contracted with three vectors can be written by

$$\breve{\mathcal{B}}_{\mu\nu\rho}\xi_1^\mu\xi_2^\nu\xi_3^\rho = \mathring{B}_{\alpha\beta\gamma}\mathcal{C}^\alpha{}_\mu\xi_1^\mu\mathcal{C}^\beta{}_\nu\xi_2^\nu\mathcal{C}^\gamma{}_\rho\xi_3^\rho = d\mathring{B}(\mathcal{C}\xi_1,\mathcal{C}\xi_2,\mathcal{C}\xi_3)\,. \tag{11.98}$$

where $\mathring{B}_{\mu\nu\rho}$ is defined in (11.40). The right-hand side of the previous relation is expressed in a non coordinate notation, which using (4.11) can be further transformed by

$$\begin{aligned}(d\mathring{B})(\mathcal{C}\xi_1,\mathcal{C}\xi_2,\mathcal{C}\xi_3) &= \mathcal{C}\xi_1\Big(\mathring{B}(\mathcal{C}\xi_2,\mathcal{C}\xi_3)\Big) - \mathcal{C}\xi_2\Big(\mathring{B}(\mathcal{C}\xi_1,\mathcal{C}\xi_3)\Big) + \mathcal{C}\xi_3\Big(\mathring{B}(\mathcal{C}\xi_1,\mathcal{C}\xi_2)\Big) \qquad (11.99)\\ &\quad -\mathring{B}\Big([\mathcal{C}\xi_1,\mathcal{C}\xi_2]_L,\mathcal{C}\xi_3\Big) + \mathring{B}\Big([\mathcal{C}\xi_1,\mathcal{C}\xi_3]_L,\mathcal{C}\xi_2\Big) - \mathring{B}\Big([\mathcal{C}\xi_2,\mathcal{C}\xi_3]_L,\mathcal{C}\xi_1\Big)\\ &= \mathcal{C}\xi_1\Big(\hat{B}(\xi_2,\xi_3)\Big) - \mathcal{C}\xi_2\Big(\hat{B}(\xi_1,\xi_3)\Big) + \mathcal{C}\xi_3\Big(\hat{B}(\xi_1,\xi_2)\Big)\\ &\quad -\hat{B}\Big([\xi_1,\xi_2]_{\hat{L}},\xi_3\Big) + \hat{B}\Big([\xi_1,\xi_3]_{\hat{L}},\xi_2\Big) - \hat{B}\Big([\xi_2,\xi_3]_{\hat{L}},\xi_1\Big)\\ &= \hat{d}\hat{B}(\xi_1,\xi_2,\xi_3)\,,\end{aligned}$$

where $\hat{B}$ is a new field that we defined by

$$\hat{B}_{\mu\nu} = \mathring{B}_{\alpha\beta}\mathcal{C}^\alpha{}_\mu\mathcal{C}^\beta{}_\nu = (B\mathcal{S}\mathcal{C})_{\mu\nu}\,, \tag{11.100}$$

and in the last step recognized the expression for twisted exterior derivative acting on a 2-form.

11.6.3 Twisted Koszul bracket

We define the twisted Koszul bracket by

$$[\lambda_1,\lambda_2]_{\breve{\theta}} = \hat{\mathcal{L}}_{\breve{\theta}(\lambda_1)}\lambda_2 - \hat{\mathcal{L}}_{\breve{\theta}(\lambda_2)}\lambda_1 - \hat{d}(\breve{\theta}(\lambda_1,\lambda_2))\,, \tag{11.101}$$

where $\breve{\theta}(\lambda_1)^\mu = \lambda_{1\nu}\breve{\theta}^{\nu\mu}$. This is an analogous definition to the one for the (non-twisted) Koszul bracket. In some local basis, its components are given by

$$\Big([\lambda_1,\lambda_2]_{\breve{\theta}}\Big)_\mu = \breve{\theta}^{\nu\rho}(\lambda_{1\nu}\hat{\partial}_\rho\lambda_{2\mu} - \lambda_{2\nu}\hat{\partial}_\rho\lambda_{1\mu}) + \hat{\partial}_\mu\breve{\theta}^{\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + (\breve{f}_{\mu\nu}{}^\rho\,\breve{\theta}^{\nu\sigma} - \breve{f}_{\mu\nu}{}^\sigma\,\breve{\theta}^{\nu\rho})\lambda_{1\rho}\lambda_{2\sigma}\,. \tag{11.102}$$

This bracket can be related to the twisted Lie bracket. In order to do that, we firstly obtain

$$\begin{aligned}\breve{\theta}\Big([\lambda_1,\lambda_2]_{\breve{\theta}}\Big) &= \breve{\theta}^{\nu\mu}\Big(\breve{\theta}^{\sigma\rho}(\lambda_{1\sigma}\hat{\partial}_\rho\lambda_{2\nu} - \lambda_{2\sigma}\hat{\partial}_\rho\lambda_{1\nu}) + \hat{\partial}_\nu\breve{\theta}^{\sigma\rho}\lambda_{1\sigma}\lambda_{2\rho} \qquad (11.103)\\ &\quad +(\breve{f}_{\nu\tau}{}^\rho\,\breve{\theta}^{\tau\sigma} - \breve{f}_{\nu\tau}{}^\sigma\,\breve{\theta}^{\tau\rho})\lambda_{1\rho}\lambda_{2\sigma}\Big)\,..\end{aligned}$$

Next, we obtain

$$
\begin{aligned}
[\breve{\theta}(\lambda_1), \breve{\theta}(\lambda_2)]_{\hat{L}} &= \breve{\theta}^{\nu\rho}\lambda_{1\rho}\hat{\partial}_\nu(\breve{\theta}^{\mu\sigma}\lambda_{2\sigma}) - \breve{\theta}^{\nu\rho}\lambda_{2\rho}\hat{\partial}_\nu(\breve{\theta}^{\mu\sigma}\lambda_{1\sigma}) + \breve{f}_{\nu\rho}^{\ \ \mu}\,\breve{\theta}^{\nu\sigma}\breve{\theta}^{\rho\tau}\lambda_{1\sigma}\lambda_{2\tau} \qquad (11.104)\\
&= \breve{\theta}^{\nu\rho}\breve{\theta}^{\mu\sigma}\Big(\lambda_{1\rho}\hat{\partial}_\nu\lambda_{2\sigma} - \lambda_{2\rho}\hat{\partial}_\nu\lambda_{1\sigma}\Big) + \breve{f}_{\nu\rho}^{\ \ \mu}\,\breve{\theta}^{\nu\sigma}\breve{\theta}^{\rho\tau}\lambda_{1\sigma}\lambda_{2\tau}\\
&\quad +(\breve{\theta}^{\rho\nu}\hat{\partial}_\nu\breve{\theta}^{\sigma\mu} + \breve{\theta}^{\sigma\nu}\hat{\partial}_\nu\breve{\theta}^{\mu\rho})\lambda_{1\rho}\lambda_{2\sigma}\,.
\end{aligned}
$$

After relabeling of some dummy indices, we obtain the relation

$$
\Big[\breve{\theta}\Big([\lambda_1, \lambda_2]_{\breve{\theta}}\Big)\Big]^\mu = \Big([\breve{\theta}(\lambda_1), \breve{\theta}(\lambda_2)]_{\hat{L}}\Big)^\mu + \breve{R}^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,, \tag{11.105}
$$

where $\breve{R}$ is defined in (11.64). We can use the definition of the twisted Lie bracket (11.89), to rewrite the previous relation as

$$
\Big[\mathcal{C}\breve{\theta}\Big([\lambda_1, \lambda_2]_{\breve{\theta}}\Big)\Big]^\mu = \Big([\mathcal{C}\breve{\theta}(\lambda_1), \mathcal{C}\breve{\theta}(\lambda_2)]_L\Big)^\mu + \mathcal{C}^\mu_{\ \sigma}\breve{R}^{\sigma\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,. \tag{11.106}
$$

The twisted Koszul bracket defines a quasi-Lie algebroid with anchor $\hat{\rho}_{\breve{\theta}} = \mathcal{C}\breve{\theta}$ and with the $\breve{R}$-flux as deformation from the Lie algebroid structure.

We can still define the exterior derivative associated with the quasi-Lie algebroid defined with the twisted Koszul bracket. On functions, its action is obtained from (5.7)

$$
\hat{d}_{\breve{\theta}}f(\lambda) = \breve{\theta}^{\mu\nu}\hat{\partial}_\nu f\lambda_\mu\,, \qquad (\hat{d}_{\breve{\theta}}f)^\mu = \breve{\theta}^{\mu\nu}\hat{\partial}_\nu f\,. \tag{11.107}
$$

Similarly, on vectors it becomes

$$
\begin{aligned}
\hat{d}_{\breve{\theta}}\xi(\lambda_1, \lambda_2) &= (\lambda_{1\rho}\breve{\theta}^{\rho\nu})\hat{\partial}_\nu(\xi^\mu\lambda_{2\mu}) - (\lambda_{2\rho}\breve{\theta}^{\rho\nu})\hat{\partial}_\nu(\xi^\mu\lambda_{1\mu}) - \xi^\mu\Big([\lambda_1, \lambda_2]_{\breve{\theta}}\Big)_\mu \qquad (11.108)\\
&= \Big(\breve{\theta}^{\mu\rho}\hat{\partial}_\rho\xi^\nu - \breve{\theta}^{\nu\rho}\hat{\partial}_\rho\xi^\mu - \xi^\rho(\hat{\partial}_\rho\breve{\theta}^{\mu\nu} + \breve{f}_{\rho\sigma}^{\ \ \mu}\,\breve{\theta}^{\sigma\nu} - \breve{f}_{\rho\sigma}^{\ \ \nu}\,\breve{\theta}^{\sigma\mu})\Big)\lambda_{1\mu}\lambda_{2\nu}\,.
\end{aligned}
$$

The exterior derivative $\hat{d}_{\breve{\theta}}$ satisfies Leibniz rule, but is not idempotent, unless the $\breve{R}$-flux is zero.

11.6.4 Twisted Schouten-Nijenhuis bracket

Lastly, we are going to interpret the term containing the $\breve{R}$-flux in terms of newly defined quasi-Lie algebroid. Recall that in the previous chapters, we defined R-flux as the Schouten-Nijenhuis bracket, that can be written as $d_\theta\theta = [\theta, \theta]_S$. This motivates us to consider the action of exterior derivative $\hat{d}_{\breve{\theta}}$ on the bi-vector $\breve{\theta}$. From definition (5.7), we have

$$
\begin{aligned}
\hat{d}_{\breve{\theta}}\breve{\theta}(\lambda_1, \lambda_2, \lambda_3) &= \mathcal{C}\breve{\theta}(\lambda_1)\Big([\lambda_2, \lambda_3]_{\breve{\theta}}\Big) - \mathcal{C}\breve{\theta}(\lambda_2)\Big([\lambda_1, \lambda_3]_{\breve{\theta}}\Big) + \mathcal{C}\breve{\theta}(\lambda_3)\Big([\lambda_1, \lambda_2]_{\breve{\theta}}\Big)\\
&\quad -\breve{\theta}\Big([\lambda_1, \lambda_2]_{\breve{\theta}}, \lambda_3\Big) + \breve{\theta}\Big([\lambda_1, \lambda_3]_{\breve{\theta}}, \lambda_2\Big) - \breve{\theta}\Big([\lambda_2, \lambda_3]_{\breve{\theta}}, \lambda_1\Big)\,. \qquad (11.109)
\end{aligned}
$$

There are two types of terms, so let us calculate the components of a representative of each type. Firstly, we have

$$\begin{aligned}\mathcal{C}\breve{\theta}(\lambda_1)\Big([\lambda_2,\lambda_3]_{\breve{\theta}}\Big) &= \lambda_{1\mu}\breve{\theta}^{\mu\sigma}\hat{\partial}_\sigma\Big(\breve{\theta}^{\nu\rho}\lambda_{2\nu}\lambda_{3\rho}\Big) \\ &= \breve{\theta}^{\mu\sigma}\hat{\partial}_\sigma\breve{\theta}^{\nu\rho}\lambda_{1\mu}\lambda_{2\nu}\lambda_{3\rho}+\breve{\theta}^{\mu\sigma}\breve{\theta}^{\nu\rho}\lambda_{1\mu}(\hat{\partial}_\sigma\lambda_{2\nu}\lambda_{3\rho}+\lambda_{2\nu}\hat{\partial}_\sigma\lambda_{3\rho})\,,\end{aligned} \tag{11.110}$$

and secondly

$$\begin{aligned}-\breve{\theta}\Big([\lambda_1,\lambda_2]_{\breve{\theta}},\lambda_3\Big) &= \lambda_{1\mu}\lambda_{2\nu}\lambda_{3\rho}\Big(\breve{\theta}^{\rho\sigma}\hat{\partial}_\sigma\breve{\theta}^{\mu\nu}+\breve{f}_{\sigma\tau}^{\ \ \mu}\,\breve{\theta}^{\nu\sigma}\breve{\theta}^{\rho\tau}-\breve{f}_{\sigma\tau}^{\ \ \nu}\,\breve{\theta}^{\mu\sigma}\breve{\theta}^{\rho\tau}\Big) \\ &\quad -\breve{\theta}^{\mu\rho}\breve{\theta}^{\nu\sigma}(\lambda_{1\nu}\hat{\partial}_\sigma\lambda_{2\mu}-\lambda_{2\nu}\hat{\partial}_\sigma\lambda_{1\mu})\lambda_{3\rho}\,.\end{aligned} \tag{11.111}$$

When (11.110) and (11.111) are substituted in (11.109), we obtain

$$\hat{d}_{\breve{\theta}}\breve{\theta}(\lambda_1,\lambda_2,\lambda_3)=2\breve{R}^{\mu\nu\rho}\lambda_{1\mu}\lambda_{2\nu}\lambda_{3\rho}\,, \tag{11.112}$$

which is exactly what we hoped for. We will therefore define the twisted Schouten-Nijenhuis bracket as

$$[\breve{\theta},\breve{\theta}]_{\hat{S}}=\hat{d}_{\breve{\theta}}\breve{\theta}\,. \tag{11.113}$$

11.7 Courant algebroid

We are now able to express all the terms in the Courant bracket twisted by B and θ in terms of newly defined twisted brackets. In the coordinate free notation, the expression (11.85) is given by

$$\begin{aligned}\xi &= [\xi_1,\xi_2]_{\hat{L}}-\kappa\breve{\theta}\Big(\hat{\mathcal{L}}_{\xi_1}\lambda_2-\hat{\mathcal{L}}_{\xi_2}\lambda_1-\frac{1}{2}\hat{d}(i_{\xi_1}\lambda_2-i_{\xi_2}\lambda_1)\Big) \\ &\quad +[\xi_1,\kappa\breve{\theta}(\lambda_2)]_{\hat{L}}-[\xi_2,\kappa\breve{\theta}(\lambda_1)]_{\hat{L}}+\frac{\kappa^2}{2}[\breve{\theta},\breve{\theta}]_{\hat{S}}(\lambda_1,\lambda_2,.) \\ &\quad +2\kappa\breve{\theta}\;\hat{d}\hat{B}(.,\xi_1,\xi_2)-2\wedge^2\kappa\breve{\theta}\;\hat{d}\hat{B}(.,\lambda_1,\xi_2)+2\wedge^2\kappa\breve{\theta}\;\hat{d}\hat{B}(.,\lambda_2,\xi_1)+2\wedge^3\kappa\breve{\theta}\;\hat{d}\hat{B}(\lambda_1,\lambda_2,.)\,,\end{aligned} \tag{11.114}$$

and the expression (11.86) by

$$\begin{aligned}\lambda &= \hat{\mathcal{L}}_{\xi_1}\lambda_2-\hat{\mathcal{L}}_{\xi_2}\lambda_1+\frac{1}{2}\hat{d}(i_{\xi_1}\lambda_2-i_{\xi_2}\lambda_1)+\kappa[\lambda_1,\lambda_2]_{\breve{\theta}} \\ &\quad +2\hat{d}\hat{B}(\xi_1,\xi_2,.)-2\kappa\breve{\theta}\;\hat{d}\hat{B}(\lambda_2,.,\xi_1)+2\kappa\breve{\theta}\;\hat{d}\hat{B}(\lambda_1,.,\xi_2)+2\wedge^2\kappa\breve{\theta}\;\hat{d}\hat{B}(\lambda_1,\lambda_2,.)\,.\end{aligned} \tag{11.115}$$

The exponents on the wedge represent how many times a bi-vector is contracted with a 3-form, while the dot denotes the non-contracted index, e.g.

$$\Big(\wedge^2\kappa\breve{\theta}\;\hat{d}\hat{B}(.,\lambda_1,\xi_2)\Big)^\mu=\kappa^2\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}\lambda_{1\nu}\xi_2^\rho\,. \tag{11.116}$$

The Courant bracket twisted at the same time by B and θ defines a Courant algebroid. The anchor is obtained from substituting $e^{-\breve{B}}$ (11.26) into (8.8)

$$\rho^{(\breve{B})}\Lambda = \mathcal{C}\xi - \kappa\mathcal{C}\breve{\theta}\lambda\,, \tag{11.117}$$

and similarly the differential operator from (11.18) and (8.10)

$$\mathcal{D}^{(\breve{B})}f = \begin{pmatrix} \kappa\mathcal{C}^{\mu}{}_{\rho}\breve{\theta}^{\rho\nu}\partial_\nu f \\ (\mathcal{C}^T)_{\mu}{}^{\nu}\partial_\nu f \end{pmatrix} = \begin{pmatrix} \hat{d}_{\breve{\theta}}f \\ \hat{d}f \end{pmatrix}\,. \tag{11.118}$$

Let us now obtain the Dirac structures for this Courant algebroid. Firstly, consider an isotropic space in the form of graph of $\mathring{B}$ over the tangent bundle

$$\mathcal{V}_{\mathring{B}}(\Lambda) = \xi^\mu \oplus 2\mathring{B}_{\mu\nu}\xi^\nu\,. \tag{11.119}$$

On this sub-bundle, the symmetry generator (11.30) becomes

$$\begin{aligned} \breve{\mathcal{G}}_{\mathcal{V}_{\mathring{B}}(\Lambda)} &= \int d\sigma \xi^\nu (\mathcal{C}_\nu{}^\mu - 2\kappa\mathring{B}_{\mu\nu}(\mathcal{S}\theta)^{\nu\rho})\pi_\mu \\ &= \int d\sigma \pi_\mu (\mathcal{C}^{-1})^\mu{}_\rho\Big((\mathcal{C}^2)^\rho{}_\nu - 2\kappa(\mathcal{S}^2\theta B)^\rho{}_\nu\Big)\xi^\nu \\ &= \int d\sigma \pi_\mu (\mathcal{C}^{-1})^\mu{}_\nu \xi^\nu\,, \end{aligned} \tag{11.120}$$

where we firstly used (11.35) and (11.8), and then (11.21). This is the generator of diffeomorphisms with the parameter $(\mathcal{C}^{-1})^\mu{}_\nu\xi^\nu$, which closes on the Lie bracket in the Poisson bracket algebra. Therefore, the sub-bundle $\mathcal{V}_{\mathring{B}}$ will be a Dirac structure and no restrictions on the $\mathring{B}$-field have to be imposed.

Similarly, we seek Dirac structures in the form of graphs of $\breve{\theta}$, i.e.

$$\mathcal{V}_{\breve{\theta}}(\Lambda) = \breve{\theta}^{\mu\nu}\lambda_\nu \oplus \lambda_\mu\,. \tag{11.121}$$

The generator (11.30) becomes

$$\breve{\mathcal{G}}_{\mathcal{V}_{\breve{\theta}}(\Lambda)} = \int d\sigma \lambda_\mu \mathcal{C}^\mu{}_\nu \kappa x'^\nu\,, \tag{11.122}$$

We encountered this case at the end of the previous chapter - this is a generator that does not depend on π and therefore gives zero bracket in its Poisson bracket algebra. The graph $\mathcal{V}_{\breve{\theta}}$ will be a Dirac structure, regardless of $\breve{\theta}$. Once again, we do not need to impose any restrictions on fluxes on the Dirac structures. Therefore, the Courant bracket twisted at the same time by B and θ defines a Courant algebroid, such that on its Dirac structures all fluxes can exist without restrictions.

Part IV

Double theory

Chapter 12

Double theory action

In this chapter, we will introduce the basic notions of a string theory defined in a phase space that is a direct sum of the initial and T-dual phase space, which we call double theory. We will introduce the Lagrangian of the double theory, derive its canonical momenta and Hamiltonian, and extend the Poisson bracket relations to the double phase space.

12.1 Lagrangian and Hamiltonian in double formalism

The idea behind the double theory [75, 76, 77, 78, 79] is the unification of the D-dimensional initial and its corresponding T-dual theory into a single theory defined in $2D$ dimensions. This theory should incorporate T-duality as its symmetry, and both the initial and T-dual theory should be obtained after projection to a suitable D-coordinate subspace. One of the most straightforward justifications for the double theory may be observed in the scenario of a closed string in which certain dimensions are compactified, enabling it to wrap around these compact dimensions. The winding number, which is to say the number of times a string curls around the compactified dimension, can be associated with the set of T-dual momenta, as it was previously demonstrated. The coordinates conjugate to these T-dual momenta y_μ are additional degrees of freedom, so the full description of the string theory should incorporate them as well.

In order to write Lagrangian, we firstly define a double coordinate X^M, defined in a direct sum of the initial coordinate space, characterized by x^μ, and T-dual coordinate space, characterized by y_μ, i.e.

$$X^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix} , \tag{12.1}$$

where $\mu = 0, 1...D-1,\ M = 0, 1...2D-1,\ D = 26$. We assume that the generalized metric has the same form as in the single theory, but with all fields, in general, depending on the double set of coordinates x^μ and y_μ

$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu}(x,y) & -2B_{\mu\rho}(x,y)(G^{-1})^{\rho\nu}(x,y) \\ 2(G^{-1})^{\mu\rho}(x,y)B_{\rho\nu}(x,y) & (G^{-1})^{\mu\nu}(x,y) \end{pmatrix} . \tag{12.2}$$

The Lagrangian density is taken in the same form as in the initial theory

$$\mathcal{L} = \frac{\kappa}{2}\partial_+ X^M H_{MN}\partial_- X^N . \tag{12.3}$$

The equations of motions from the variation of the Lagrangian (12.3) become

$$\partial_+(H_{MN}\partial_- X^N) + \partial_-(H_{MN}\partial_+ X^N) = 0 . \tag{12.4}$$

In the case of constant background fields, these relations simplify to

$$\partial_+\partial_- x^\mu = 0 , \quad \partial_+\partial_- y_\mu = 0 , \tag{12.5}$$

which are well-known equations of motion for the initial and T-dual theories, respectively. The relation (12.4) is also known as the Bianchi identity. We see that the Bianchi identities and equations of motion are united into a single relation in double formalism.

The double set of coordinates is accompanied by the double set of momenta conjugate to them. It can be easily obtained by varying the Lagrangian (12.3) with respect to $\dot{X}^M$

$$\Pi_M = \frac{\delta\mathcal{L}}{\delta\dot{X}^M} = \kappa H_{MN}\dot{X}^N , \tag{12.6}$$

which can be written in the component notation as

$$\Pi_M = \begin{pmatrix} \pi_\mu \\ {}^\star\pi^\mu \end{pmatrix} , \tag{12.7}$$

where

$$\pi_\mu = G^E_{\mu\nu}\dot{x}^\nu - 2(BG^{-1})_\mu{}^\nu\dot{y}_\nu , \tag{12.8}$$

and

$${}^\star\pi^\mu = (G^{-1})^{\mu\nu}\dot{y}_\nu + 2(G^{-1}B)^\mu{}_\nu\dot{x}^\nu . \tag{12.9}$$

We can easily obtain the inverse of the relation (12.6)

$$\dot{X}^M = \frac{1}{\kappa}H^{MN}\Pi_N , \tag{12.10}$$

where H^{MN} is the inverse of the generalized metric, given by

$$H^{MN} = \eta^{MK}H_{KL}\eta^{LN} . \tag{12.11}$$

Now we can apply the Legendre transformation of the Lagrangian (12.3), in order to obtain the canonical Hamiltonian

$$\mathcal{H}_{\mathcal{C}} = \Pi_M \dot{X}^M - \mathcal{L} = \frac{1}{2\kappa}\Pi_M H^{MN}\Pi_N + \frac{\kappa}{2}X'^M H_{MN} X'^N \,, \tag{12.12}$$

where we used (12.10).

12.2 T-duality

Our goal is to rewrite the Buscher T-duality transformation laws (3.22) and (3.25) in the double formalism. We note that by separating terms that change sign with those that do not, the T-duality relations can be rewritten as

$$\begin{aligned} \pm\partial_\pm y_\mu &\simeq G^E_{\mu\nu}\partial_\pm x^\nu - 2(BG^{-1})_\mu{}^\nu \partial_\pm y_\nu \,, \\ \pm\partial_\pm x^\mu &\simeq 2(G^{-1}B)^\mu{}_\nu \partial_\pm x^\nu + (G^{-1})^{\mu\nu}\partial_\pm y_\nu \,, \end{aligned} \tag{12.13}$$

which can be easily integrated into a single relation

$$\partial_\pm X^M \simeq \pm\eta^{MN} H_{NK}\, \partial_\pm X^K \,. \tag{12.14}$$

To obtain the canonical form of T-duality relations, using (2.19) we rewrite (12.14)

$$\dot{X}^M \pm X'^M \simeq \eta^{MN} H_{NK} X'^K \pm \eta^{MN} H_{NK}\dot{X}^K \,, \tag{12.15}$$

or equivalently

$$\dot{X}^M \simeq \eta^{MN} H_{NK} X'^K \,, \quad X'^M \simeq \eta^{MN} H_{NK}\dot{X}^K \,, \tag{12.16}$$

which using (12.6) can be expressed as

$$\Pi_M \simeq \kappa\, \eta_{MN} X'^M \,. \tag{12.17}$$

After application of (12.14) to (12.3), we have

$$\frac{\kappa}{2}\partial_+ X^M H_{MN}\partial_- X^N \simeq -\frac{\kappa}{2}\eta^{MK} H_{KL}\partial_+ X^L H_{MN}\eta^{NP} H_{PQ}\partial_- X^Q = -\mathcal{L} \,. \tag{12.18}$$

The Lagrangian is (up to a sign) invariant under T-duality. The change of sign does not matter, since equations of the motion will remain the same, and we will have exactly the same theory.

12.3 Poisson bracket relations in double formalism

As double theory should generalize both initial and T-dual theory, we assume the standard Poisson bracket relations within the initial and T-dual phase spaces

$$\{x'^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta'(\sigma - \bar\sigma)\,, \qquad \{y'_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = \delta^\nu_\mu \delta'(\sigma - \bar\sigma)\,, \tag{12.19}$$

with other brackets of canonical variables within the same phase space being zero, i.e.

$$\{\kappa x'^\mu(\sigma), \kappa x'^\nu(\bar\sigma)\} = \{\kappa y'_\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} = 0\,, \tag{12.20}$$

and

$$\{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = \{{}^\star\pi^\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = 0\,. \tag{12.21}$$

These relations have to be extended so that they include relations of phase space variables from mutually T-dual phase spaces, which we will do using T-duality. Let us firstly apply T-duality along all coordinates y_μ to the Poisson bracket relation between coordinate derivatives in mutually T-dual phase spaces

$$\{\kappa x'^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} \simeq \{\kappa x'^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \kappa \delta^\mu_\nu \delta'(\sigma - \bar\sigma)\,. \tag{12.22}$$

Relations (12.20) and (12.22) can be rewritten in terms of double coordinates as

$$\{\kappa X'^M(\sigma), \kappa X'^N(\bar\sigma)\} \simeq \kappa\, \eta_{MN} \delta'(\sigma - \bar\sigma)\,. \tag{12.23}$$

If we were to obtain the Poisson bracket relation between double coordinates, rather than their derivatives, we could integrate the previous relation along both σ and $\bar\sigma$, and obtain

$$\{\kappa X^M(\sigma), \kappa X^N(\bar\sigma)\} = -\kappa \eta^{MN} \theta(\sigma - \bar\sigma)\,, \tag{12.24}$$

where θ is Heavyside step function. The relation (12.24) is determined up to boundary conditions, that can be set with different choice of Heavyside step function.

Secondly, we apply T-dualization along all y_μ coordinates to the Poisson bracket of momenta from mutually T-dual spaces, and obtain

$$\{\pi_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \simeq \kappa \{\pi_\mu(\sigma), x'^\nu(\bar\sigma)\} = \kappa \delta^\nu_\mu \delta'(\sigma - \bar\sigma)\,. \tag{12.25}$$

Relations (12.21) and (12.25) nicely combine into

$$\{\Pi_M(\sigma), \Pi_N(\bar\sigma)\} \simeq \kappa\, \eta_{MN} \delta'(\sigma - \bar\sigma)\,. \tag{12.26}$$

Lastly, we once again T-dualize along all the initial coordinates x^μ to obtain the remaining bracket

$$\{\kappa x'^\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \simeq \{{}^\star\pi^\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = 0\,, \tag{12.27}$$

which with the other brackets (12.19) can be written as

$$\{X'^M(\sigma), \Pi_N(\bar{\sigma})\} = \delta^M_N \delta'(\sigma - \bar{\sigma})\,. \tag{12.28}$$

Some Poisson bracket relations are written as T-duality relations, emphasizing that double theory intrinsically incorporates T-duality.

12.4 Restricted fields

While it is true that background fields depend on both initial and T-dual coordinates, in order to achieve invariance under both diffeomorphisms and T-dual diffeomorphisms, specific constraints must be imposed to the background fields. Firstly, we will demand that all fields are annihilated by the operator

$$\Delta = \eta^{MN}\partial_M\partial_N = \partial^M\partial_M = 0\,, \tag{12.29}$$

where ∂^M are the derivatives in a double theory, given by

$$\partial_M = \begin{pmatrix} \partial_\mu \\ \tilde{\partial}^\mu \end{pmatrix}\,, \qquad \left(\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,, \quad \tilde{\partial}^\mu \equiv \frac{\partial}{\partial y_\mu}\right)\,. \tag{12.30}$$

Moreover, we will require the so-called strong constraints, in which all the products of any two fields ϕ and ψ are also annihilated by (12.29), i.e.

$$\partial^M\partial_M(\phi\psi) = (\partial^M\partial_M\phi)\,\psi + 2\partial^M\phi\,\partial_M\psi + \phi\,\partial^M\partial_M\psi = 2\partial^M\phi\,\partial_M\psi = 0\,. \tag{12.31}$$

These conditions appear also from the Virasoro conditions [80, 81]. Without strong constraints, the symmetry algebra would not close.

Chapter 13

C-bracket

In this chapter, we provide the world-sheet derivation of the C-bracket, which is the double theory generalization of the Lie bracket. We will present the double generator of diffeomorphisms and show that the C-bracket appears in its algebra. We will end this chapter by considering the projection of the C-bracket to the initial and T-dual phase spaces and show that it reduces to the Courant bracket.

13.1 Generator of diffeomorphisms in double theory

Previously, we saw that the diffeomorphisms are generated by momenta π_μ, and we expect that T-dual diffeomorphisms are generated by T-dual momenta ${}^\star\pi^\mu$. In double theory, these momenta are integrated into a double momentum Π_M (12.7). We will construct the double generator which is a sum of generators of initial and T-dual diffeomorphisms. It can be written as the $O(D,D)$ invariant inner product

$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, \Pi \rangle \,, \tag{13.1}$$

where Λ^M are the symmetry parameters, which can be expressed by

$$\Lambda^M(X) = \begin{pmatrix} \xi^\mu(x,y) \\ \lambda_\mu(x,y) \end{pmatrix} \,. \tag{13.2}$$

The parameters ξ^μ are associated with initial diffeomorphisms, while the parameters λ_μ are associated with T-dual diffeomorphisms. Both parameters depend on all initial coordinates x^μ and all T-dual coordinates y_μ.

We want to obtain the Poisson bracket relations of a double generator. We have

$$\begin{aligned} \{\mathcal{G}_{\Lambda_1}(\sigma), \mathcal{G}_{\Lambda_2}(\bar{\sigma})\} &= \int d\sigma d\bar{\sigma}\Big(\Lambda_1^M(\sigma)\{\Pi_M(\sigma), \Pi_N(\bar{\sigma})\}\Lambda_2^N(\bar{\sigma}) \\ &\quad +\Lambda_1^M(\sigma)\{\Pi_M(\sigma), \Lambda_2^N(\bar{\sigma})\}\Pi_N(\bar{\sigma}) + \Pi_M(\sigma)\{\Lambda_1^M(\sigma), \Pi_N(\bar{\sigma})\}\Lambda_2^N(\bar{\sigma})\Big)\,. \end{aligned} \tag{13.3}$$

We did not write the term $\Pi_M(\sigma)\{\Lambda_1^M(\sigma), \Lambda_2^N(\bar{\sigma})\}\Pi_N(\bar{\sigma})$, since it is zero after applying the strong constraint (12.31) and the chain rule

$$\{\Lambda_1^M(\sigma), \Lambda_2^N(\bar{\sigma})\} = -\frac{1}{\kappa}\partial^P\Lambda_1^M\partial_P\Lambda_2^N\theta(\sigma-\bar{\sigma}) = -\frac{1}{\kappa}\Big(\Delta(\Lambda_1^M\Lambda_2^N) - \Delta\Lambda_1^M\Lambda_2^N\Big)\theta(\sigma-\bar{\sigma}) = 0\,. \tag{13.4}$$

Without this condition, the generator algebra would be anomalous. In fact, there were successful attempts to find constraints that are weaker than the strong constraint that we imposed, in which case anomalous part in the algebra contributes to the trivial transformation [82]. We are primarily interested in constructing the C-bracket, and for this purpose, it is sufficient to assume strong constraints (12.31).

To the first term of (13.3), we apply the relation (12.26)

$$\kappa\Lambda_1^M(\sigma)\{\Pi_M(\sigma), \Pi_N(\bar{\sigma})\}\Lambda_2^N(\bar{\sigma})\delta'(\sigma-\bar{\sigma}) \simeq \kappa\langle\Lambda_1(\sigma), \Lambda_2(\bar{\sigma})\rangle\delta'(\sigma-\bar{\sigma})\,. \tag{13.5}$$

After applying (7.7) on the right-hand side of the previous relation, we obtain

$$\begin{aligned} \kappa\langle\Lambda_1(\sigma), \Lambda_2(\bar{\sigma})\rangle\delta'(\sigma-\bar{\sigma}) &= \frac{\kappa}{2}\Big(\langle\Lambda_1, \Lambda_2'\rangle - \langle\Lambda_1', \Lambda_2\rangle\Big)\delta(\sigma-\bar{\sigma}) \\ &\quad +\frac{\kappa}{2}\Big(\langle\Lambda_1, \Lambda_2\rangle + \langle\Lambda_1, \Lambda_2\rangle(\bar{\sigma})\Big)\delta'(\sigma-\bar{\sigma})\,, \end{aligned} \tag{13.6}$$

where parameters depend on σ unless otherwise explicitly expressed. With the help of the chain rule

$$\kappa\Lambda'^M = \kappa\, X'^N\partial_N\Lambda^M\,, \tag{13.7}$$

the relation (13.6) further transforms into

$$\begin{aligned} \kappa\langle\Lambda_1(\sigma), \Lambda_2(\bar{\sigma})\rangle\delta'(\sigma-\bar{\sigma}) = &\quad \frac{1}{2}\eta_{PQ}\left(\Lambda_1^P\partial_N\Lambda_2^Q - \Lambda_2^P\partial_N\Lambda_1^Q\right)X'^N\delta(\sigma-\bar{\sigma}) \\ &\quad +\frac{\kappa}{2}\Big(\langle\Lambda_1, \Lambda_2\rangle(\sigma) + \langle\Lambda_1, \Lambda_2\rangle(\bar{\sigma})\Big)\delta'(\sigma-\bar{\sigma})\,. \end{aligned} \tag{13.8}$$

The anomaly disappears after the integration with respect to σ and $\bar{\sigma}$. The first line in (13.8) contributes to the C-bracket expression. We apply the T-duality relations (12.17) to it, and obtain

$$\frac{1}{2}\eta_{PQ}\left(\Lambda_1^P\partial_N\Lambda_2^Q - \Lambda_2^P\partial_N\Lambda_1^Q\right)X'^N \simeq \frac{1}{2}\eta_{PQ}\,\eta^{MN}\Big(\Lambda_1^P\partial_N\Lambda_2^Q - \Lambda_2^P\partial_N\Lambda_1^Q\Big)\Pi_M\,. \tag{13.9}$$

Note that we applied T-duality twice - in (13.5) and (13.9), and consequentially we can write

$$\Lambda_1^M(\sigma)\{\Pi_M(\sigma), \Pi_N(\bar{\sigma})\}\Lambda_2^N(\bar{\sigma}) = \frac{1}{2}\eta_{PQ}\,\eta^{MN}\Big(\Lambda_1^P\partial_N\Lambda_2^Q - \Lambda_2^P\partial_N\Lambda_1^Q\Big)\Pi_M\delta(\sigma-\bar{\sigma})\,. \tag{13.10}$$

After relabeling of some dummy indices, the remaining terms in (13.3) can be written as

$$-\Pi_M(\Lambda_1^N\partial_N\Lambda_2^M - \Lambda_2^N\partial_N\Lambda_1^M)\delta(\sigma-\bar\sigma) \tag{13.11}$$

From relations (13.10) and (13.11), we can express the generator algebra relations

$$\{\mathcal{G}_{\Lambda_1}, \mathcal{G}_{\Lambda_2}\} = -\mathcal{G}_{[\Lambda_1,\Lambda_2]_{\mathbf{C}}}\,, \tag{13.12}$$

where $[\Lambda_1,\Lambda_2]_{\mathbf{C}}$ is the C-bracket, given by

$$\Big([\Lambda_1,\Lambda_2]_{\mathbf{C}}\Big)^M = \Lambda_1^N\partial_N\Lambda_2^M - \Lambda_2^N\partial_N\Lambda_1^M - \frac{1}{2}\Big(\Lambda_1^N\partial^M\Lambda_{2N} - \Lambda_2^N\partial^M\Lambda_{1N}\Big)\,. \tag{13.13}$$

The C-bracket was firstly obtained by Siegel [80, 81]. It is a generalization of the Lie bracket to double space. One can introduce the double Lie derivative $\hat{\mathcal{L}}_\Lambda$, that acts on all indices as if they were both covariant and contravariant and its algebra will give C-bracket. For example, its action on the generalized metric is given by

$$\hat{\mathcal{L}}_\Lambda H^{MN} = \Lambda^P\partial_P H^{MN} + (\partial^M\Lambda_P - \partial_P\Lambda^M)H^{PN} + (\partial^N\Lambda_P - \partial_P\Lambda^N)H^{MP}\,. \tag{13.14}$$

If no dependence on T-dual momenta and T-dual coordinates exists, the generator $\mathcal{G}_\Lambda$ would be just the generator of diffeomorphisms. Its algebra is known to close and produces the Lie bracket.

13.2 Projections to the initial and T-dual phase spaces

Let us consider projections of the C-bracket to the initial and T-dual phase spaces. Firstly, we will demand that all parameters in (13.13) depend exclusively on the initial coordinates x^μ. In that case, the double derivative ∂^M reduces to the derivative along x^μ, i.e.

$$\partial^M \to \begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix}\,. \tag{13.15}$$

The terms in the C-bracket also simplify

$$\Lambda_1^N\partial_N\Lambda_2^M \to \begin{pmatrix} \xi_1^\nu\partial_\nu\xi_2^\mu \\ \xi_1^\nu\partial_\nu\lambda_{2\mu} \end{pmatrix}\,,\quad \Lambda_1^N\partial^M\Lambda_{2N} \to \begin{pmatrix} 0 \\ \lambda_{1\nu}\partial_\mu\xi_2^\nu + \xi_1^\nu\partial_\mu\lambda_{2\nu} \end{pmatrix}\,, \tag{13.16}$$

where parameters depend only on x. Substituting the previous relation into (13.13), we obtain the projection of the C-bracket to the initial phase space

$$\begin{aligned}
[\Lambda_1,\Lambda_2]_{\mathbf{C}} \to &\begin{pmatrix} \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu \\ \xi_1^\nu\partial_\nu\lambda_{2\mu} - \xi_2^\nu\partial_\nu\lambda_{1\mu} - \frac{1}{2}(\lambda_{1\nu}\partial_\mu\xi_2^\nu - \lambda_{2\nu}\partial_\mu\xi_1^\nu + \xi_1^\nu\partial_\mu\lambda_{2\nu} - \xi_2^\nu\partial_\mu\lambda_{1\nu}) \end{pmatrix} \\
&= \begin{pmatrix} \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu \\ \xi_1^\nu(\partial_\nu\lambda_{2\mu} - \partial_\mu\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\lambda_{1\mu} - \partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1^\nu\lambda_{2\nu} - \xi_2^\nu\lambda_{1\nu}) \end{pmatrix}\,,
\end{aligned}$$

where we used the chain rule, in order to recognize the result as the Courant bracket. By projecting the C-bracket to the initial theory, we obtained the standard Courant bracket.

Secondly, by ignoring all dependence on x^μ, we will obtain the C-bracket projection to the T-dual phase space. Then, the double derivative reduces to the derivative along T-dual coordinates y_μ

$$\partial^M \to \begin{pmatrix} \tilde{\partial}^\mu \\ 0 \end{pmatrix} , \tag{13.17}$$

and similarly, we obtain

$$\Lambda_1^N \partial_N \Lambda_2^M \to \begin{pmatrix} \lambda_{1\nu} \tilde{\partial}^\nu \xi_2^\mu \\ \lambda_{1\nu} \tilde{\partial}^\nu \lambda_{2\mu} \end{pmatrix} , \quad \Lambda_1^N \partial^M \Lambda_{2N} \to \begin{pmatrix} \xi_1^\nu \tilde{\partial}^\mu \lambda_{2\nu} + \lambda_{1\nu} \tilde{\partial}^\mu \xi_2^\nu \\ 0 \end{pmatrix} . \tag{13.18}$$

Now all parameters depend solely on T-dual coordinates y_μ. Substituting (13.18) into (13.13), one obtains

$$\begin{aligned} [\Lambda_1, \Lambda_2]_{\mathbf{C}} \to & \begin{pmatrix} \lambda_{1\nu} \tilde{\partial}^\nu \xi_2^\mu - \lambda_{2\nu} \tilde{\partial}^\nu \xi_1^\mu - \frac{1}{2} (\xi_1^\nu \tilde{\partial}^\mu \lambda_{2\nu} + \lambda_{1\nu} \tilde{\partial}^\mu \xi_2^\nu - \xi_2^\nu \tilde{\partial}^\mu \lambda_{1\nu} - \lambda_{2\nu} \tilde{\partial}^\mu \xi_1^\nu) \\ \lambda_{1\nu} \tilde{\partial}^\nu \lambda_{2\mu} - \lambda_{2\nu} \tilde{\partial}^\nu \lambda_{1\mu} \end{pmatrix} \\ = & \begin{pmatrix} \lambda_{1\nu} (\tilde{\partial}^\nu \xi_2^\mu - \tilde{\partial}^\mu \xi_2^\nu) - \lambda_{2\nu} (\tilde{\partial}^\nu \xi_1^\mu - \tilde{\partial}^\mu \xi_1^\nu) + \frac{1}{2} \tilde{\partial}^\mu (\lambda_{1\nu} \xi_2^\nu - \xi_1^\nu \lambda_{2\nu}) \\ \lambda_{1\nu} \tilde{\partial}^\nu \lambda_{2\mu} - \lambda_{2\nu} \tilde{\partial}^\nu \lambda_{1\mu} \end{pmatrix} . \end{aligned}$$

We applied the chain rule in this instance as well. The resulting bracket is again the Courant bracket. This time, the symmetry parameters ξ^μ and λ_μ have swapped their roles.

Both in the initial and T-dual theory, the C-bracket reduces to the Courant bracket. This way, the invariance of the Courant bracket under T-duality is shown once again. The C-bracket is a double theory extension of the Courant bracket.

Chapter 14

B-twisted C-bracket

We are going to obtain the B-twisted C-bracket, together with its corresponding flux. Subsequently, we will consider this bracket's projection to the initial and T-dual phase space and show that in the former it produces the B-twisted Couran bracket, while in the latter, it produces the θ-twisted Courant bracket.

14.1 Non-canonical basis and basic algebra relations

The generator in the double theory also has a form of the $O(D,D)$ invariant inner product, allowing us to generalize the procedure of twisting the Courant bracket for twisting the C-bracket in the double theory. Following the path we took in Chapter 9, we define a diagonal generalized metric G_{MN} by

$$G_{MN} = \begin{pmatrix} G_{\mu\nu}(x,y) & 0 \\ 0 & (G^{-1})^{\mu\nu}(x,y) \end{pmatrix} , \tag{14.1}$$

which by the action of B-transformation produces the generalized metric H_{MN}

$$((e^{\hat{B}})^T)_M^{\ K}\, G_{KL} (e^{\hat{B}})^L_{\ N} = H_{MN} \, , \quad (e^{\hat{B}})^M_{\ N} = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu}(x,y) & \delta_\mu^\nu \end{pmatrix} . \tag{14.2}$$

The B-transformation has exact same form as we encountered before (6.5), with the only difference that the Kalb-Ramond field now depends both on the initial coordinates x^μ and the T-dual coordinates y_μ. When relation (14.2) is substituted into (12.12) we obtain the free-form expression of the canonical Hamiltonian

$$\mathcal{H}_{\mathcal{C}} = \frac{1}{2\kappa} \hat{\Pi}_M G^{MN} \hat{\Pi}_N + \frac{\kappa}{2} \hat{X}'^M G_{MN} \hat{X}'^N \, , \tag{14.3}$$

where we introduced the non-canonical momenta $\hat{\Pi}$ by

$$\hat{\Pi}^M = (e^{\hat{B}})^M{}_N \Pi^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} {}^\star\pi^\nu \\ \pi_\nu \end{pmatrix} = \begin{pmatrix} {}^\star\pi^\mu \\ \pi_\mu + 2B_{\mu\nu}{}^\star\pi^\nu \end{pmatrix} \equiv \begin{pmatrix} {}^\star\pi^\mu \\ \hat{\pi}_\mu \end{pmatrix} , \tag{14.4}$$

and non-canonical coordinates σ-derivatives $\hat{X}'$ by

$$\hat{X}'^M = (e^{\hat{B}})^M{}_N X'^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} x'^\nu \\ y'_\nu \end{pmatrix} = \begin{pmatrix} x'^\mu \\ y'_\mu + 2B_{\mu\nu}x'^\nu \end{pmatrix} \equiv \begin{pmatrix} x'^\mu \\ \hat{y}'_\mu \end{pmatrix} . \tag{14.5}$$

The generator $\mathcal{G}_\Lambda$ (13.1) can be expressed in terms of non-canonical momenta $\hat{\Pi}$ by

$$\hat{\mathcal{G}}_{\hat{\Lambda}} = \int d\sigma \langle \hat{\Lambda}, \hat{\Pi} \rangle , \tag{14.6}$$

where we introduced a new symmetry parameter $\hat{\Lambda}$, related to the parameter Λ (13.2) by

$$\hat{\Lambda}^M = (e^{\hat{B}})^M{}_N \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \lambda_\mu + 2B_{\mu\nu}\xi^\nu \end{pmatrix} \equiv \begin{pmatrix} \xi^\mu \\ \hat{\lambda}_\mu \end{pmatrix} . \tag{14.7}$$

Using the fact that $e^{\hat{B}}$ is an $O(D,D)$ transformation, the generator algebra (13.12) when expressed in terms of generator $\hat{\mathcal{G}}_\Lambda$ takes the form

$$\{\hat{\mathcal{G}}_{\hat{\Lambda}_1}(\sigma), \hat{\mathcal{G}}_{\hat{\Lambda}_2}(\bar{\sigma})\} = -\hat{\mathcal{G}}_{[\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathbf{C}_B}}(\sigma)\delta(\sigma-\bar{\sigma}) , \tag{14.8}$$

where $[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathbf{C}_B}$ we define as the B-twisted C-bracket, given by

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathbf{C}_B} = e^{\hat{B}}[e^{-\hat{B}}\hat{\Lambda}_1, e^{-\hat{B}}\hat{\Lambda}_2]_{\mathbf{C}} . \tag{14.9}$$

In order to obtain the B-twisted C-bracket from the Poisson bracket algebra, we require the Poisson bracket relations between non-canonical momenta $\hat{\Pi}$. Using (14.4) we write

$$\begin{aligned} \{\hat{\Pi}^M(\sigma), \hat{\Pi}^N(\bar{\sigma})\} &= \{(e^{\hat{B}}\Pi)^M(\sigma), (e^{\hat{B}}\Pi)^N(\bar{\sigma})\} \\ &= (e^{\hat{B}})^M_J(\sigma)(e^{\hat{B}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \\ &\quad -(e^{\hat{B}})^M_J\partial^J(e^{\hat{B}})^N_K\Pi^K\delta(\sigma-\bar{\sigma}) + (e^{\hat{B}})^N_J\partial^J(e^{\hat{B}})^M_K\Pi^K\delta(\sigma-\bar{\sigma}) . \end{aligned} \tag{14.10}$$

Next, using the T-duality relations (12.26) on the first term of the right-hand side of the previous expression, we obtain

$$(e^{\hat{B}})^M_J(\sigma)(e^{\hat{B}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \simeq \kappa \left[e^B(\sigma)\eta(e^B)^T(\bar{\sigma})\right]^{MN}\delta'(\sigma-\bar{\sigma}) , \tag{14.11}$$

which can be further tranformed by

$$\Big[e^{B}(\sigma)\eta(e^{B})^T(\bar\sigma)\Big]^{MN}\delta'(\sigma-\bar\sigma) = \kappa\eta^{MN}\delta'(\sigma-\bar\sigma) + \kappa(e^{\hat B})^M_{\ P}\eta^{PR}\partial_Q((e^{\hat B})^T)_R^{\ N}X'^Q\delta(\sigma-\bar\sigma)\,, \tag{14.12}$$

where we have used (7.7) and (6.3) for B-shifts. After applying the T-dual relations (12.17) to the non-anomalous part of (14.12), we obtain

$$\kappa(e^{\hat B})^M_{\ P}\,\eta^{PR}\partial_Q((e^{\hat B})^T)_R^{\ N}X'^Q\delta(\sigma-\bar\sigma) \simeq (e^{\hat B})^M_{\ P}\,\partial_Q\hat B^{PN}\Pi^Q\delta(\sigma-\bar\sigma)\,. \tag{14.13}$$

We note the following properties of matrix $\hat B^M_{\ N}$ (6.5)

$$\hat B^M{}_K\hat B^K{}_N = 0\,,\qquad \hat B^M{}_K\partial^Q\hat B^K{}_N = 0\,,\qquad (e^{\hat B})^M{}_N = \delta^M{}_N + \hat B^M{}_N\,, \tag{14.14}$$

and rewrite the relation (14.10) as

$$\{\hat\Pi^M(\sigma),\hat\Pi^N(\bar\sigma)\} = -\hat B^{MNQ}\,\hat\Pi_Q\delta(\sigma-\bar\sigma) + A^{MN}(\sigma-\bar\sigma)\,. \tag{14.15}$$

With A^{MN} we have marked the anomalous term, given by

$$A^{MN}(\sigma-\bar\sigma) \simeq \kappa\eta^{MN}\delta'(\sigma-\bar\sigma)\,, \tag{14.16}$$

and with $\hat B^{MNQ}$ the double flux, where

$$\begin{aligned} \hat B^{MNQ} &= B^{MNQ} + S^{MNQ} \\ B^{MNQ} &= \partial^M\hat B^{NQ} + \partial^N\hat B^{QM} + \partial^Q\hat B^{MN} \\ S^{MNQ} &= \hat B^M{}_K\partial^K\hat B^{NQ} + \hat B^N{}_K\partial^K\hat B^{QM} + \hat B^Q{}_K\partial^K\hat B^{MN}\,. \end{aligned} \tag{14.17}$$

Flux can be written in a more compact manner

$$\hat B^{MNQ} = \hat\partial^M\hat B^{NQ} + \hat\partial^N\hat B^{QM} + \hat\partial^Q\hat B^{MN}\,, \tag{14.18}$$

where $\hat\partial$ is a new double derivative, given by

$$\hat\partial^M = (e^{\hat B})^M{}_K\partial^K = \partial^M + \hat B^M_{\ K}\,\partial^K\,. \tag{14.19}$$

Appart from the relation (14.15), we will also need another basic Poisson relation

$$\{\hat\Lambda^M(\sigma),\hat\Pi^N(\bar\sigma)\} = \hat\partial^N\hat\Lambda^M\delta(\sigma-\bar\sigma)\,, \tag{14.20}$$

and note that the bracket between parameters is zero, i.e.

$$\{\hat\Lambda^M(\sigma),\hat\Lambda^N(\bar\sigma)\} = 0\,, \tag{14.21}$$

as a direct consequence of (12.29) and (12.31) (see discussion below (13.4) for more details).

14.2 Derivation of B-twisted C-bracket

Substituting (14.15), (14.20) and (14.21) into (14.8), we obtain

$$\begin{aligned}\{\hat{\mathcal{G}}_{\hat{\Lambda}_1}(\sigma), \hat{\mathcal{G}}_{\hat{\Lambda}_2}(\bar{\sigma})\} &= \hat{\Lambda}_1^M(\sigma)\hat{\Lambda}_2^N(\bar{\sigma})A_{MN} - \hat{\Lambda}_{1M}\hat{\Lambda}_{2N}\hat{B}^{MNQ}\hat{\Pi}_Q\delta(\sigma-\bar{\sigma}) \\ &+\hat{\Pi}_Q\Big[\hat{\Lambda}_2^N\hat{\partial}_N\hat{\Lambda}_1^Q - \hat{\Lambda}_1^N\hat{\partial}_N\hat{\Lambda}_2^Q\Big]\delta(\sigma-\bar{\sigma})\,.\end{aligned} \tag{14.22}$$

The first term containing anomaly is transformed with the help of (7.7) and (14.16) by

$$\begin{aligned}&\hat{\Lambda}_1^M(\sigma)\hat{\Lambda}_2^N(\bar{\sigma})A_{MN}(\sigma-\bar{\sigma}) \\ &\simeq \kappa\langle\hat{\Lambda}_1(\sigma),\hat{\Lambda}_2(\sigma)\rangle\delta'(\sigma-\bar{\sigma}) + \kappa\langle\hat{\Lambda}_1(\sigma),\hat{\Lambda}_2'(\sigma)\rangle\delta(\sigma-\bar{\sigma}) \\ &= \frac{\kappa}{2}\Big(2\langle\hat{\Lambda}_1,\hat{\Lambda}_2\rangle\delta'(\sigma-\bar{\sigma}) + \langle\hat{\Lambda}_1,\hat{\Lambda}_2\rangle'\delta(\sigma-\bar{\sigma})\Big) + \frac{\kappa}{2}\Big(\langle\hat{\Lambda}_1,\hat{\Lambda}_2'\rangle - \langle\hat{\Lambda}_1',\hat{\Lambda}_2\rangle\Big)\delta(\sigma-\bar{\sigma}) \\ &= \frac{\kappa}{2}\Big(\langle\hat{\Lambda}_1,\hat{\Lambda}_2\rangle(\sigma) + \langle\hat{\Lambda}_1,\hat{\Lambda}_2\rangle(\bar{\sigma})\Big)\delta'(\sigma-\bar{\sigma}) + \frac{\kappa}{2}\Big(\langle\hat{\Lambda}_1,\hat{\Lambda}_2'\rangle - \langle\hat{\Lambda}_1',\hat{\Lambda}_2\rangle\Big)\delta(\sigma-\bar{\sigma})\,,\end{aligned} \tag{14.23}$$

resulting in two terms. The first term is anomalous and dissappears after the integration. On the second term, the T-duality relations (12.17) can be applied, after which one obtains

$$\begin{aligned}\frac{\kappa}{2}\Big(\langle\hat{\Lambda}_1,\hat{\Lambda}_2'\rangle - \langle\hat{\Lambda}_1',\hat{\Lambda}_2\rangle\Big) &= \frac{\kappa}{2}\eta_{MN}\Big(\hat{\Lambda}_1^M\partial_Q\hat{\Lambda}_2^N - \hat{\Lambda}_2^M\partial_Q\hat{\Lambda}_1^N\Big)X'^Q \\ &\simeq \frac{1}{2}\eta_{MN}\Big(\hat{\Lambda}_1^M\partial^Q\hat{\Lambda}_2^N - \hat{\Lambda}_2^M\partial^Q\hat{\Lambda}_1^N\Big)\Pi_Q \\ &= \frac{1}{2}\eta_{MN}\Big(\hat{\Lambda}_1^M\hat{\partial}^Q\hat{\Lambda}_2^N - \hat{\Lambda}_2^M\hat{\partial}^Q\hat{\Lambda}_1^N\Big)\hat{\Pi}_Q\,,\end{aligned} \tag{14.24}$$

where we used (14.19) and (14.4). Note that this is a second application of T-duality, which acts as equality. The substitution of (14.23) and (14.24) into (14.22) results in the final expression for the B-twisted C-bracket

$$\begin{aligned}\Big([\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathbf{C}_B}\Big)^M &= \hat{\Lambda}_1^N\hat{\partial}_N\hat{\Lambda}_2^M - \hat{\Lambda}_2^N\hat{\partial}_N\hat{\Lambda}_1^M \\ &-\frac{1}{2}\Big(\hat{\Lambda}_1^N\hat{\partial}^M\hat{\Lambda}_{2N} - \hat{\Lambda}_2^N\hat{\partial}^M\hat{\Lambda}_{1N}\Big) + \hat{\Lambda}_{1N}\hat{\Lambda}_{2Q}\hat{B}^{MNQ}\,.\end{aligned} \tag{14.25}$$

With the substitution of (14.19) into its expression, B-twisted C-bracket becomes

$$\begin{aligned}\Big([\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathbf{C}_B}\Big)^M &= \hat{\Lambda}_1^N\partial_N\hat{\Lambda}_2^M - \hat{\Lambda}_2^N\partial_N\hat{\Lambda}_1^M - \frac{1}{2}\Big(\hat{\Lambda}_1^N\partial^M\hat{\Lambda}_{2N} - \hat{\Lambda}_2^N\partial^M\hat{\Lambda}_{1N}\Big) \\ &+\hat{B}^N_{\ R}\Big(\hat{\Lambda}_{1N}\partial^R\hat{\Lambda}_2^M - \hat{\Lambda}_{2N}\partial^R\hat{\Lambda}_1^M\Big) - \frac{1}{2}\hat{B}^M_{\ R}\Big(\hat{\Lambda}_{1N}\partial^R\hat{\Lambda}_2^N - \hat{\Lambda}_{2N}\partial^R\hat{\Lambda}_1^N\Big) \\ &+\hat{\Lambda}_{1N}\hat{\Lambda}_{2Q}\hat{B}^{MNQ}\,.\end{aligned} \tag{14.26}$$

We can see that the first line is the C-bracket, while the other two lines are corrections due to twisting. If the Kalb-Ramond field is zero, the second and the third lines become zero and the bracket reduces to the C-bracket.

14.3 Projections to the initial and T-dual phase space

Firstly, let us consider the B-twisted C-bracket projected to the initial phase space. It can be obtained by demanding that all gauge fields depend solely on the initial coordinates x^μ. In that case, the derivatives $\hat{\partial}^M$ become just derivatives along the initial coordinates x^μ

$$\hat{\partial}^M \to \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \partial_\nu \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix} . \tag{14.27}$$

The terms from the bracket simplify as

$$\hat{\Lambda}_1^N \hat{\partial}_N \hat{\Lambda}_2^M \to \begin{pmatrix} \xi_1^\nu \partial_\nu \xi_2^\mu \\ \xi_1^\nu \partial_\nu \hat{\lambda}_{2\mu} \end{pmatrix} , \tag{14.28}$$

and

$$\hat{\Lambda}_1^N \hat{\partial}^M \hat{\Lambda}_{2N} \to \begin{pmatrix} 0 \\ \hat{\lambda}_{1\nu} \partial_\mu \xi_2^\nu + \xi_1^\nu \partial_\mu \hat{\lambda}_{2\nu} \end{pmatrix} , \tag{14.29}$$

while the flux $\hat{B}^{MNQ}$ reduces to the standard H-flux, i.e.

$$\hat{B}^{MNQ} \hat{\Lambda}_{1N} \hat{\Lambda}_{2Q} \to \begin{pmatrix} 0 \\ 2B_{\mu\nu\rho} \xi_1^\nu \xi_2^\rho \end{pmatrix} . \tag{14.30}$$

Combining previous relations and using the chain rule, we obtain

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathbf{C}_B} \to [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_B} = \hat{\Lambda} \equiv \begin{pmatrix} \xi \\ \hat{\lambda} \end{pmatrix} , \tag{14.31}$$

where

$$\begin{aligned} \xi^\mu &= \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu , \\ \hat{\lambda}_\mu &= \xi_1^\nu (\partial_\nu \hat{\lambda}_{2\mu} - \partial_\mu \hat{\lambda}_{2\nu}) - \xi_2^\nu (\partial_\nu \hat{\lambda}_{1\mu} - \partial_\mu \hat{\lambda}_{1\nu}) + \frac{1}{2} \partial_\mu (\xi_1 \hat{\lambda}_2 - \xi_2 \hat{\lambda}_1) + 2B_{\mu\nu\rho} \xi_1^\nu \xi_2^\rho . \end{aligned} \tag{14.32}$$

The B-twisted C-bracket becomes B-twisted Courant bracket in the initial theory.

Secondly, let us obtain the projection of B-twisted C-bracket to the T-dual phase space, by demanding that all variables depend solely on T-dual coordinates y_μ. In this case, the derivative $\hat{\partial}^M$ becomes

$$\hat{\partial}^M \to \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \tilde{\partial}^\nu \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\partial}^\mu \\ 2B_{\mu\nu} \tilde{\partial}^\nu \end{pmatrix} , \tag{14.33}$$

so that the bracket terms reduce to

$$\hat{\Lambda}_1^N \partial_N \hat{\Lambda}_2^M \to \begin{pmatrix} \hat{\lambda}_{1\nu}\tilde{\partial}^\nu \xi_2^\mu + 2B_{\nu\rho}\xi_1^\rho \tilde{\partial}^\nu \xi_2^\mu \\ \hat{\lambda}_{1\nu}\tilde{\partial}^\nu \hat{\lambda}_{2\mu} + 2B_{\nu\rho}\xi_1^\rho \tilde{\partial}^\nu \hat{\lambda}_{2\mu} \end{pmatrix}, \tag{14.34}$$

and

$$\hat{\Lambda}_1^N \hat{\partial}^M \hat{\Lambda}_{2N} \to \begin{pmatrix} \hat{\lambda}_{1\nu}\tilde{\partial}^\mu \xi_2^\nu + \xi_1^\nu \tilde{\partial}^\mu \hat{\lambda}_{2\nu} \\ 2\hat{\lambda}_{1\nu} B_{\mu\rho}\tilde{\partial}^\rho \xi_2^\nu + 2\xi_1^\nu B_{\mu\rho}\tilde{\partial}^\rho \hat{\lambda}_{2\nu} \end{pmatrix}. \tag{14.35}$$

The term containing flux $\hat{B}^{MNQ}$ becomes

$$\hat{B}^{MNQ}\hat{\Lambda}_{1N}\hat{\Lambda}_{2Q} \to \begin{pmatrix} \kappa\, {}^\star Q^\mu{}_{\nu\rho}\xi_1^\nu \xi_2^\rho \\ \kappa\, {}^\star Q^\nu{}_{\rho\mu}(\xi_1^\rho \hat{\lambda}_{2\nu} - \xi_2^\rho \hat{\lambda}_{1\nu}) + \kappa^{2\star} R_{\mu\nu\rho}\xi_1^\nu \xi_2^\rho \end{pmatrix}, \tag{14.36}$$

where we have marked the non-geometric fluxes in T-dual theory as a function of the T-dual non-commutative parameter ${}^\star\theta_{\mu\nu} = \frac{2}{\kappa} B_{\mu\nu}$ by

$$\kappa\, {}^\star Q^\mu{}_{\nu\rho} = 2\tilde{\partial}^\mu B_{\nu\rho} = \kappa\, \tilde{\partial}^\mu\, {}^\star\theta_{\nu\rho}\,, \tag{14.37}$$

and

$$\begin{aligned} \kappa^{2\star} R_{\mu\nu\rho} &= 4B_{\mu\sigma}\tilde{\partial}^\sigma B_{\nu\rho} + 4B_{\nu\sigma}\tilde{\partial}^\sigma B_{\rho\mu} + 4B_{\rho\sigma}\tilde{\partial}^\sigma B_{\mu\nu}\,, \\ &= \kappa^2\, {}^\star\theta_{\mu\sigma}\tilde{\partial}^\sigma\, {}^\star\theta_{\nu\rho} + \kappa^2\, {}^\star\theta_{\nu\sigma}\tilde{\partial}^\sigma\, {}^\star\theta_{\rho\mu} + \kappa^2\, {}^\star\theta_{\rho\sigma}\tilde{\partial}^\sigma\, {}^\star\theta_{\mu\nu}\,. \end{aligned} \tag{14.38}$$

Combining previous relations and using the chain rule, the B-twisted C-bracket becomes

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathbf{C}_B} \to [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_B} = \hat{\Lambda} \equiv \begin{pmatrix} \xi \\ \hat{\lambda} \end{pmatrix}, \tag{14.39}$$

where

$$\begin{aligned} \xi^\mu &= \hat{\lambda}_{1\nu}(\tilde{\partial}^\nu \xi_2^\mu - \tilde{\partial}^\mu \xi_2^\nu) - \hat{\lambda}_{2\nu}(\tilde{\partial}^\nu \xi_1^\mu - \tilde{\partial}^\mu \xi_1^\nu) + \tilde{\partial}^\mu(\xi_1 \hat{\lambda}_2 - \xi_2 \hat{\lambda}_1) \\ &\quad 2B_{\nu\rho}(\xi_1^\nu \tilde{\partial}^\rho \xi_2^\mu - \xi_2^\nu \tilde{\partial}^\rho \xi_1^\mu) + 2\tilde{\partial}^\mu B_{\nu\rho}\xi_1^\nu \xi_2^\rho\,, \\ \hat{\lambda}_\mu &= \hat{\lambda}_{1\nu}\tilde{\partial}^\nu \hat{\lambda}_{2\mu} - \hat{\lambda}_{2\nu}\tilde{\partial}^\nu \hat{\lambda}_{1\mu} \\ &\quad -2B_{\mu\nu}\Big(\hat{\lambda}_{1\rho}(\tilde{\partial}^\nu \xi_2^\rho - \tilde{\partial}^\rho \xi_2^\nu) - \hat{\lambda}_{2\rho}(\tilde{\partial}^\nu \xi_1^\rho - \tilde{\partial}^\rho \xi_1^\nu) - \frac{1}{2}\tilde{\partial}^\nu(\hat{\lambda}_1 \xi_2 - \hat{\lambda}_2 \xi_1)\Big) \\ &\quad +2\hat{\lambda}_{1\nu}\tilde{\partial}^\nu(\xi_2^\rho B_{\rho\mu}) - 2\hat{\lambda}_{2\nu}\tilde{\partial}^\nu(\xi_1^\rho B_{\rho\mu}) + 2(\xi_1^\nu B_{\nu\rho})\tilde{\partial}^\rho \hat{\lambda}_{2\mu} - 2(\xi_2^\nu B_{\nu\rho})\tilde{\partial}^\rho \hat{\lambda}_{1\mu} \\ &\quad +4\Big(B_{\mu\sigma}\tilde{\partial}^\sigma B_{\nu\rho} + B_{\nu\sigma}\tilde{\partial}^\sigma B_{\rho\mu} + B_{\rho\sigma}\tilde{\partial}^\sigma B_{\mu\nu}\Big)\xi_1^\nu \xi_2^\rho\,. \end{aligned} \tag{14.40}$$

With the following change of variables

$${}^\star\hat{\lambda}^\mu = \xi^\mu\,, \quad {}^\star\xi_\mu = \hat{\lambda}_\mu\,, \tag{14.41}$$

the B-twisted C-bracket becomes ${}^{\star}\theta$-twisted Courant bracket in the T-dual phase space (14.40).

These are very interesting results that show the attractiveness of the double theory. When we considered only the initial theory and symmetries therein, we needed to act with different transformations on the generator's basis to obtain the B- and θ-twisted Courant bracket. However, in double theory, they are both easily obtained from the projection of the B-twisted C-bracket to the relevant subspaces. Moreover, these projections reduce the double flux $\hat{B}^{MNQ}$ to the geometric H-flux, and also non-geometric Q and R-fluxes, depending on the phase space to which we project it. Lastly, we discussed T-duality in the context of isomorphism between Courant algebroids and showed that with the exchange of background fields with their T-duals according to the Buscher rules, together with momenta and coordinate σ-derivatives, B-twisted and θ-twisted Courant algebroids transform into each other. This Courant algebroid isomorphism becomes manifest in a double theory, as both algebroids can be obtained from the single bracket defined in a double space.

Chapter 15

θ-twisted C-bracket

This chapter we devote to the derivation of the θ-twisted C-bracket and its corresponding double flux. They have the same form as their B-twisted counterparts, which was not the case with their respective Courant brackets. We consider the projection of this bracket to the mutually T-dual phase spaces and obtain the θ-twisted C-bracket in the initial, and B-twisted C-bracket in the T-dual phase space.

15.1 Non-canonical basis and basic algebra relations

In the analogy with the derivation of the θ-twisted Courant bracket, we consider the string moving in a double space-time characterized by the T-dual metric

$$^\star G_{MN} = \begin{pmatrix} ^\star G^{-1}_{\mu\nu}(x,y) & 0 \\ 0 & ^\star G^{\mu\nu}(x,y) \end{pmatrix} = \begin{pmatrix} G^E_{\mu\nu}(x,y) & 0 \\ 0 & (G_E^{-1})^{\mu\nu}(x,y) \end{pmatrix} , \tag{15.1}$$

where G_E is defined in (2.36). The generalized metric can be obtained from the action of θ-transformation $e^{\hat\theta}$ (6.7)

$$^\star H_{MN} = ((e^{\hat\theta})^T)_M^{\ L}\, {}^\star G_{LK} (e^{\hat\theta})^K_{\ N} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix} , \tag{15.2}$$

which is exactly equal to the generalized metric (2.38), with the difference of background fields depending also on T-dual coordinates y_μ. In terms of $^\star G_{MN}$ (15.1), the canonical Hamiltonian (12.12) is written in the form of a free Hamiltonian as

$$\mathcal{H}_\mathcal{C} = \frac{1}{2\kappa}\breve\Pi_M {}^\star G^{MN} \breve\Pi_N + \frac{\kappa}{2}\breve X'^M {}^\star G_{MN} \breve X'^N , \tag{15.3}$$

where the new non-canonical double coordinates σ-derivatives are given by

$$\breve X'^M = (e^{\hat\theta})^M_{\ N} X'^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} x'^\nu \\ y'_\nu \end{pmatrix} = \begin{pmatrix} x'^\mu + \kappa\theta^{\mu\nu} y'_\nu \\ y'_\mu \end{pmatrix} \equiv \begin{pmatrix} \breve x'^\mu \\ y'_\mu \end{pmatrix} , \tag{15.4}$$

and new non-canonical double momenta by

$$\breve{\Pi}^M = (e^{\hat{\theta}})^M{}_N \Pi^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} {}^\star\pi^\nu \\ \pi_\nu \end{pmatrix} = \begin{pmatrix} {}^\star\pi^\mu + \kappa\theta^{\mu\nu}\pi_\nu \\ \pi_\mu \end{pmatrix} \equiv \begin{pmatrix} {}^\star\breve{\pi}^\mu \\ \pi_\mu \end{pmatrix} . \tag{15.5}$$

In this basis, the symmetry generator is given by

$$\breve{\mathcal{G}}_{\breve{\Lambda}} = \int d\sigma \langle \breve{\Lambda}, \breve{\Pi} \rangle \,, \tag{15.6}$$

where

$$\breve{\Lambda}^M = (e^{\hat{\theta}})^M{}_N \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu + \kappa\theta^{\mu\nu}\lambda_\nu \\ \lambda_\mu \end{pmatrix} \equiv \begin{pmatrix} \breve{\xi}^\mu \\ \lambda_\mu \end{pmatrix} . \tag{15.7}$$

The θ-twisted C-bracket appears in the algebra of generators (15.6), via relation

$$\{\breve{\mathcal{G}}_{\breve{\Lambda}_1}(\sigma), \breve{\mathcal{G}}_{\breve{\Lambda}_2}(\bar{\sigma})\} = -\breve{\mathcal{G}}_{[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta}}(\sigma)\delta(\sigma - \bar{\sigma}) \,, \tag{15.8}$$

where

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta} = e^{\hat{\theta}}[e^{-\hat{\theta}}\breve{\Lambda}_1, e^{-\hat{\theta}}\breve{\Lambda}_2]_{\mathbf{C}} \,. \tag{15.9}$$

In order to compute this bracket, we need to obtain the algebra between non-canonical momenta, which is expanded as

$$\begin{aligned} \{\breve{\Pi}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} &= \{(e^{\hat{\theta}}\Pi)^M(\sigma), (e^{\hat{\theta}}\Pi)^N(\bar{\sigma})\} \\ &= (e^{\hat{\theta}})^M_J(\sigma)(e^{\hat{\theta}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \\ &\quad -(e^{\hat{\theta}})^M_J \partial^J (e^{\hat{\theta}})^N_Q \Pi^Q \delta(\sigma - \bar{\sigma}) + (e^{\hat{\theta}})^N_J \partial^J (e^{\hat{\theta}})^M_Q \Pi^Q \delta(\sigma - \bar{\sigma}) \,. \end{aligned} \tag{15.10}$$

Using (12.26) and (7.7), we obtain

$$(e^{\hat{\theta}})^M_J(\sigma)(e^{\hat{\theta}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} = A^{MN}(\sigma - \bar{\sigma}) + (e^{\hat{\theta}})^M_P \partial_Q \hat{\theta}^{PN} \Pi^Q \delta(\sigma - \bar{\sigma}) \,, \tag{15.11}$$

where A^{MN} is the same anomaly defined in (14.16). Substituting (15.11) and (15.5) into (15.10), we obtain

$$\{\breve{\Pi}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} = -\breve{\Theta}^{MNQ}\breve{\Pi}_Q \delta(\sigma - \bar{\sigma}) + A^{MN}(\sigma - \bar{\sigma}) \,, \tag{15.12}$$

where

$$\begin{aligned} \breve{\Theta}^{MNQ} &= \Theta^{MNQ} + R^{MNQ} \\ \Theta^{MNQ} &= \partial^M \hat{\theta}^{NQ} + \partial^N \hat{\theta}^{QM} + \partial^Q \hat{\theta}^{MN} \\ R^{MNQ} &= \hat{\theta}^M{}_K \partial^K \hat{\theta}^{NQ} + \hat{\theta}^N{}_K \partial^K \hat{\theta}^{QM} + \hat{\theta}^Q{}_K \partial^K \hat{\theta}^{MN} \,. \end{aligned} \tag{15.13}$$

In a similar manner as when twisting by B, we introduce derivatives $\breve{\partial}^M$ by

$$\breve{\partial}^M = (e^{\hat{\theta}})^M_{\ N}\, \partial^N = \partial^M + \hat{\theta}^M_{\ N}\, \partial^N \,, \tag{15.14}$$

and express the flux in a more compact form

$$\breve{\Theta}^{MNR} = \breve{\partial}^M \hat{\theta}^{NR} + \breve{\partial}^N \hat{\theta}^{RM} + \breve{\partial}^R \hat{\theta}^{MN} \,. \tag{15.15}$$

From definition of $\breve{\Pi}^M$ one easily obtains the relation

$$\{\breve{\Lambda}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} = \breve{\partial}^N \breve{\Lambda}^M \delta(\sigma - \bar{\sigma}) \,, \tag{15.16}$$

and from the strong constraints (12.31), the algebra between symmetry parameters is zero.

We note that the algebra relations between non-canonical momenta $\hat{\Pi}^M$ (14.15) and parameters $\hat{\Lambda}^M$ (14.20) on the one side, and non-canonical momenta $\breve{\Pi}^M$ (15.12) and parameters $\breve{\Lambda}^M$ (15.16) on the other side, have the exact same form. The difference is that the former basic relations are expressed in terms of derivatives $\hat{\partial}^M$ (14.19) and flux $\hat{B}^{MNR}$ (14.18), and the latter in terms of derivatives $\breve{\partial}^M$ (15.14) and flux $\breve{\Theta}^{MNR}$ (15.15). Therefore, the θ-twisted C-bracket can be obtained from relation (14.25), simply by substituting the relevant expressions with their analogons. We obtain

$$\begin{aligned} \Big([\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta}\Big)^M &= \breve{\Lambda}_1^N \breve{\partial}_N \breve{\Lambda}_2^M - \breve{\Lambda}_2^N \breve{\partial}_N \breve{\Lambda}_1^M \\ &\quad -\frac{1}{2}\Big(\breve{\Lambda}_1^N \breve{\partial}^M \breve{\Lambda}_{2N} - \breve{\Lambda}_2^N \breve{\partial}^M \breve{\Lambda}_{1N}\Big) + \breve{\Lambda}_{1N} \breve{\Lambda}_{2Q} \breve{\Theta}^{MNQ} \,, \end{aligned} \tag{15.17}$$

which once the expression (15.14) is substituted becomes

$$\begin{aligned} \Big([\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta}\Big)^M &= \breve{\Lambda}_1^N \partial_N \breve{\Lambda}_2^M - \breve{\Lambda}_2^N \partial_N \breve{\Lambda}_1^M - \frac{1}{2}\Big(\breve{\Lambda}_1^N \partial^M \breve{\Lambda}_{2N} - \breve{\Lambda}_2^N \partial^M \breve{\Lambda}_{1N}\Big) \\ &\quad +\hat{\theta}^N_{\ R}\Big(\breve{\Lambda}_{1N} \partial^R \breve{\Lambda}_2^M - \breve{\Lambda}_{2N} \partial^R \breve{\Lambda}_1^M\Big) - \frac{1}{2}\hat{\theta}^M_{\ R}\Big(\breve{\Lambda}_{1N} \partial^R \breve{\Lambda}_2^N - \breve{\Lambda}_{2N} \partial^R \breve{\Lambda}_1^N\Big) \\ &\quad +\breve{\Lambda}_{1N} \breve{\Lambda}_{2Q} \breve{\Theta}^{MNQ} \,. \end{aligned} \tag{15.18}$$

The first line is the C-bracket, while the remaining terms are contributions due to its twisting by θ.

15.2 Projections to the initial and T-dual phase space

We conclude this chapter with the projections of the θ-twisted C-bracket to the initial and T-dual phase spaces. In the former case, all fields and parameters will only depend on the initial coordinates x^μ. The derivative $\breve{\partial}^M$ becomes

$$\breve{\partial}^M \to \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \partial_\nu \end{pmatrix} = \begin{pmatrix} \kappa\theta^{\mu\nu}\partial_\nu \\ \partial_\mu \end{pmatrix} \,, \tag{15.19}$$

and moreover

$$\breve{\Lambda}_1^N \breve{\partial}_N \breve{\Lambda}_2^M \to \begin{pmatrix} \lambda_{1\nu}\kappa\theta^{\nu\rho}\partial_\rho \breve{\xi}_2^\mu + \breve{\xi}_1^\nu \partial_\nu \breve{\xi}_2^\mu \\ \lambda_{1\nu}\kappa\theta^{\nu\rho}\partial_\rho \lambda_{2\mu} + \breve{\xi}_1^\nu \partial_\nu \lambda_{2\mu} \end{pmatrix} , \tag{15.20}$$

and

$$\breve{\Lambda}_1^N \breve{\partial}^M \breve{\Lambda}_{2N} \to \begin{pmatrix} \kappa\theta^{\mu\nu}(\breve{\xi}_1^\rho \partial_\nu \lambda_{2\rho} + \lambda_{1\rho}\partial_\nu \breve{\xi}_2^\rho) \\ \breve{\xi}_1^\rho \partial_\mu \lambda_{2\rho} + \lambda_{1\rho}\partial_\mu \breve{\xi}_2^\rho \end{pmatrix} . \tag{15.21}$$

The flux term is given by

$$\breve{\Lambda}_{1N}\breve{\Lambda}_{2Q}\breve{\Theta}^{MNQ} \to \begin{pmatrix} \kappa^2 R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + \kappa Q_\nu^{\ \rho\mu}(\breve{\xi}_1^\nu \lambda_{2\rho} - \breve{\xi}_2^\nu \lambda_{1\rho}) \\ \kappa Q_\mu^{\ \rho\nu}\lambda_{1\rho}\lambda_{2\nu} \end{pmatrix} , \tag{15.22}$$

where Q and R are non-geometric fluxes (10.8).

Substituting (15.20), (15.21) and (15.22) into (15.17) we obtain the projection of the θ-twisted C-bracket

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta} \to [\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathcal{C}_\theta} = \breve{\Lambda} \equiv \begin{pmatrix} \breve{\xi} \\ \lambda \end{pmatrix} , \tag{15.23}$$

where

$$\begin{aligned}
\breve{\xi}^\mu \;&=\; \breve{\xi}_1^\nu \partial_\nu \breve{\xi}_2^\mu - \breve{\xi}_2^\nu \partial_\nu \breve{\xi}_1^\mu + \\
&\quad -\kappa\theta^{\mu\nu}\Big(\breve{\xi}_1^\rho(\partial_\nu\lambda_{2\rho} - \partial_\rho\lambda_{2\nu}) - \breve{\xi}_2^\rho(\partial_\nu\lambda_{1\rho} - \partial_\rho\lambda_{1\nu}) - \frac{1}{2}\partial_\nu(\breve{\xi}_1\lambda_2 - \breve{\xi}_2\lambda_1)\Big) \\
&\quad +\kappa\breve{\xi}_1^\nu\partial_\nu(\lambda_{2\rho}\theta^{\rho\mu}) - \kappa\breve{\xi}_2^\nu\partial_\nu(\lambda_{1\rho}\theta^{\rho\mu}) + \kappa(\lambda_{1\nu}\theta^{\nu\rho})\partial_\rho\breve{\xi}_2^\mu - \kappa(\lambda_{2\nu}\theta^{\nu\rho})\partial_\rho\breve{\xi}_1^\mu \\
&\quad +\kappa^2(\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu})\lambda_{1\nu}\lambda_{2\rho} , \\
\lambda_\mu \;&=\; \breve{\xi}_1^\nu(\partial_\nu\lambda_{2\mu} - \partial_\mu\lambda_{2\nu}) - \breve{\xi}_2^\nu(\partial_\nu\lambda_{1\mu} - \partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\breve{\xi}_1\lambda_2 - \breve{\xi}_2\lambda_1) \\
&\quad +\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu} - \lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}\partial_\mu\theta^{\rho\nu} .
\end{aligned} \tag{15.24}$$

These are relations defining the θ-twisted Courant bracket.

On the other hand, the projection to the T-dual phase space is obtained by keeping only the terms with the T-dual coordinates y_μ. The double derivatives are just derivatives along the T-dual coordinates y_μ, i.e.

$$\breve{\partial}^M \to \begin{pmatrix} \delta_\nu^\mu & \kappa\theta^{\mu\nu} \\ 0 & \delta_\mu^\nu \end{pmatrix} \begin{pmatrix} \tilde{\partial}^\nu \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\partial}^\mu \\ 0 \end{pmatrix} . \tag{15.25}$$

Furthermore, we have

$$\breve{\Lambda}_1^N \breve{\partial}_N \breve{\Lambda}_2^M \to \begin{pmatrix} \lambda_{1\nu}\tilde{\partial}^\nu \breve{\xi}_2^\mu \\ \lambda_{1\nu}\tilde{\partial}^\nu \lambda_{2\mu} \end{pmatrix} \tag{15.26}$$

and

$$\breve{\Lambda}_1^N \breve{\partial}^M \breve{\Lambda}_{2N} \to \begin{pmatrix} \lambda_{1\nu} \tilde{\partial}^\mu \breve{\xi}_2^\nu + \breve{\xi}_1^\nu \tilde{\partial}^\mu \lambda_{2\nu} \\ 0 \end{pmatrix} , \tag{15.27}$$

while the flux term is simply given by

$$\breve{\Lambda}_{1N} \breve{\Lambda}_{2Q} \breve{\Theta}^{MNQ} \to \begin{pmatrix} \kappa \, {}^\star B^{\mu\nu\rho} \lambda_{1\nu} \lambda_{2\rho} \\ 0 \end{pmatrix} , \tag{15.28}$$

where ${}^\star B^{\mu\nu\rho}$ is the H flux in T-dual theory

$$\begin{aligned} \kappa {}^\star B^{\mu\nu\rho} &= \kappa \, \tilde{\partial}^\mu \theta^{\nu\rho} + \kappa \, \tilde{\partial}^\nu \theta^{\rho\mu} + \kappa \, \tilde{\partial}^\rho \theta^{\mu\nu} \\ &= 2\tilde{\partial}^\mu \, {}^\star B^{\nu\rho} + 2\tilde{\partial}^\nu \, {}^\star B^{\rho\mu} + 2\tilde{\partial}^\rho \, {}^\star B^{\mu\nu} \, . \end{aligned} \tag{15.29}$$

The expression for θ-twisted C-bracket projected to the T-dual phase space is given by

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta} \to [\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathcal{C}_\theta} = \breve{\Lambda} \equiv \begin{pmatrix} \breve{\xi} \\ \lambda \end{pmatrix} , \tag{15.30}$$

where

$$\begin{aligned} \breve{\xi}^\mu &= \lambda_{1\nu} (\tilde{\partial}^\nu \breve{\xi}_2^\mu - \tilde{\partial}^\mu \breve{\xi}_2^\nu) - \lambda_{2\nu} (\tilde{\partial}^\nu \breve{\xi}_1^\mu - \tilde{\partial}^\mu \breve{\xi}_1^\nu) + \frac{1}{2} \tilde{\partial}^\mu (\breve{\xi}_1 \lambda_2 - \breve{\xi}_2 \lambda_1) \\ &\quad + \kappa \, {}^\star B^{\mu\nu\rho} \lambda_{1\nu} \lambda_{2\rho} \, , \\ \lambda_\mu &= \lambda_{1\nu} \tilde{\partial}^\nu \lambda_{2\mu} - \lambda_{2\nu} \tilde{\partial}^\nu \lambda_{1\mu} \, . \end{aligned} \tag{15.31}$$

This is the Courant bracket twisted by a 2-form ${}^\star B$.

In the case of θ-twisted C-bracket, we see that both the Courant bracket twisted by B and by θ can be obtained from it, depending on which phase space we project. The isomorphism between two Courant algebroids appears naturally as a T-duality transformation between different projections of the bracket, in the same way as in the case of B-twisted C-bracket. It also features the flux that in different projections contains both H-flux and non-geometric Q and R fluxes.

Part V

Conclusions

In this thesis, we considered the application of generalized geometry to bosonic string theory and obtained various Courant algebroid structures in the symmetry algebra relations. Primarily we focused on the algebroid brackets and their properties, obtaining different fluxes in the algebroid relations. Moreover, we established relations of these brackets with T-duality, in both the single and double theory approach. For each algebroid, we obtained its Dirac structures and the constraints they impose on string fluxes.

The bosonic string σ-model is invariant under two groups of symmetries - diffeomorphisms and local gauge transformations. The generators of these transformations are self T-dual, so we united these generators into a single generator. It can be expressed as the $O(D, D)$ invariant inner product of two generalized vectors, one of which is the double symmetry parameter, a direct sum of diffeomorphism and local gauge transformations parameters, and another one is the double canonical variable, a direct sum of the coordinate σ derivative and canonical momenta. We obtained the Poisson bracket algebra relations of these generators and showed that it closes on another generator parametrized with the Courant bracket of two double symmetry parameters. The Courant bracket is a well-known bracket on the generalized tangent bundle. We showed that it is in fact the self T-dual extension of the Lie bracket. The Courant bracket defines the standard Courant algebroid, which consists of the generalized tangent bundle as its vector bundle, the $O(D, D)$ invariant inner product, and the projection to the tangent bundle as its anchor. The Dirac structures related to the standard Courant algebroid are a symplectic manifold and a Poisson manifold. Translated into the language of string fluxes, these are spaces in which the H-flux and R-flux have to be zero, respectively.

Afterward, we developed a method of obtaining the twisted Courant bracket by an arbitrary $O(D, D)$ transformation. The method consists of choosing a different basis in which the generator is written, obtained by the action of the $O(D, D)$ transformation on a double canonical variable. If the symmetry parameter is transformed with the same transformation, the generator will remain the same, due to it being the $O(D, D)$ invariant inner product. We demonstrated that in the Poisson bracket algebra of such a generator, the twisted Courant bracket appears. Moreover, there is a natural way to define the Courant algebroid, consisting of the generalized tangent bundle, the twisted Courant bracket, the $O(D, D)$ invariant inner product, and the anchor defined as a composition of the natural projection to the tangent bundle and the inverse of the $O(D, D)$ transformation used for twisting. We showed that all five Courant algebroid conditions are apriori satisfied.

We chose three transformations relevant to string theory and twisted the Courant bracket by them, using the aforementioned method. Firstly, we considered B-transformations and with it acted on the double canonical variable. The resulting generalized vector consists of coordinates σ derivatives as its vector and auxiliary currents i_μ as its 1-form components. This is a non-canonical basis, but when expressed in it, the Hamiltonian has a form of a free Hamiltonian, written in terms of diagonal generalized metric. The structure function of the Poisson bracket algebra of auxiliary currents is the

Kalb-Ramond field strength, i.e. the H-flux. We expressed the symmetry generator in this basis and obtained its algebra, where the Courant bracket twisted by B appeared. The bracket differs from the Courant bracket by a term containing H-flux. We obtained Dirac structures corresponding to this bracket on which an arbitrary H-flux can exist. On the other set of Dirac structures, written in the form of a graph of bi-vector over a cotangent bundle, we showed that the generalized R-flux has to be zero.

Secondly, we considered the background characterized only by the effective metric, which is the T-dual metric. We acted with the θ-transformation as a similarity transformation and obtained the generalized metric. It is possible to express this Hamiltonian in terms of the new non-canonical basis obtained with the action of the θ-transformation to the double canonical variable. The resulting basis consists of a new set of auxiliary currents k^μ and canonical momenta. In the algebra of auxiliary currents, the non-geometric $Q-$ and $R-$fluxes appear as structure functions. We obtained the θ-twisted Courant bracket in the Poisson algebra of this generator. Some of the terms in the θ-twisted Courant bracket include the Koszul bracket (the Lie bracket generalization to the cotangent bundle) and Schouten-Nijenhuis bracket (the Lie bracket generalization to the space of multivectors). We showed that on Dirac structures related to the Courant algebroid with θ-twisted Courant bracket, R-flux can exist without restrictions on the non-commutativity parameter.

We derived the B-twisted and θ-twisted Courant brackets in [1, 2]. What we found as a peculiar property is their relation via T-duality. The T-duality is a known string phenomenon where winding and momenta numbers are interchanged. The former are obtained when the coordinate σ derivative is integrated around the compact dimension, and the latter when the canonical momenta are integrated. Moreover, the non-commutativity parameter and effective metric are T-duals of the Kalb-Ramond field and metric tensor, respectively. The T-duality can be realized in the same phase space, by interchanging canonical momenta and coordinate σ derivatives, together with the interchange of background fields with their T-duals. We coined this term self T-duality and showed that it directly relates two generators - one giving rise to the B-twisted Courant bracket and another giving rise to the θ-twisted Courant bracket in the Poisson bracket algebra. Because we were working in the same phase space, we were able to obtain the coordinate transformation that takes the parameters of the one generator and results in the parameters of the other generator. We showed that this transformation defines the isomorphism between two Courant algebroids. This way, we demonstrated that B-twisted and θ-twisted Courant brackets are self T-dual.

Thirdly, we obtained the Courant bracket that was simultaneously twisted both by B and θ. This bracket was first obtained in [3]. Beforehand, only the successive twists were considered, in which case the Courant bracket twisted firstly by B and afterward by θ was obtained. This bracket, sometimes referred to as the Roytenberg bracket, contains all generalized fluxes, but the bracket itself is not invariant under T-duality. This is due to the fact that B-shifts and θ-transformations do not commute.

Instead, we considered the matrix $\breve{B}$, which is a sum of $\hat{B}$ and $\hat{\theta}$, exponents of which govern twists of the Courant bracket by B and θ, respectively. By construction, this transformation is invariant under T-duality. The price we paid is that the square of the matrix $\breve{B}$ is not zero, and therefore all terms in Taylor's expansion had to be obtained. The full twisting matrix contained hyperbolic functions of the matrix $\alpha^{\mu}{}_{\nu} = 2\kappa\theta^{\mu\rho}B_{\rho\nu}$.

Computing the $B-\theta$-twisted Courant bracket was not an easy task. On the first hand, it seemed to produce a meaningless conundrum, with the appearance of a plethora of terms with no obvious interpretation. Luckily, we were able to overcome this obstacle by considering another twist, which was related to the simultaneous twist by B and θ by a simple coordinate transformation. This auxiliary twist gave rise to the currents in a simpler form, such that it was possible to obtain the fluxes, which were then related to the fluxes of the $B-\theta$-twisted Courant bracket by an inverse of the above-mentioned twist.

We showed that this bracket contains all generalized fluxes. The H-flux is defined as a field strength of an antisymmetric field defined on the Lie algebroid, with the twisted Lie bracket as its bracket, while the R-flux we expressed as the twisted Schouten-Nijenhuis bracket of new bi-vectors $\breve{\theta}$. The bi-vector $\breve{\theta}$ is in general not the Poisson one, so it defines the quasi-Lie algebroid with the twisted Koszul bracket as its bracket. It is possible to define the non-nilpotent exterior derivative corresponding to the twisted Koszul bracket. Its action on the bi-vector $\breve{\theta}$ gives the R-flux and defines the twisted Schouten-Nijenhuis bracket. We found an interesting result when we computed the Dirac structures of the $B-\theta$-twisted Courant bracket: all generalized fluxes can exist on Dirac structures, with no restrictions imposed on them.

In the end, we generalized results to the case of double theory, in which all fields depend on both initial and T-dual coordinates. We considered diffeomorphisms, generated by canonical momenta, and T-dual diffeomorphisms, generated by T-dual canonical momenta. The parameters were taken to depend on both the initial and T-dual coordinates. We extended the Poisson bracket relations to the double space, taking into account that they should commute with T-duality relations. The generator governing both diffeomorphisms and T-dual diffeomorphisms was written in the form of an $O(D,D)$ invariant inner product. It has been shown it gives rise to the C-bracket, which was published in [1]. The C-bracket reduces to the Courant bracket when either all the initial coordinates or all the T-dual coordinates are projected out.

In addition, we twisted the C-bracket in the same way as the Courant bracket. We first considered the Hamiltonian with the generalized metric in the diagonal form, containing only metric tensor, and the Kalb-Ramond field appearing only through a flux in the non-canonical variables algebra. In its generator algebra, we obtained the B-twisted C-bracket. It extends the C-bracket with additional terms due to twisting, including the double theory flux. When dependence on T-dual coordinates is neglected, the bracket reduces to the B-twisted Courant bracket. On the other hand, when dependence on the initial coordinates is neglected, the bracket becomes the θ-twisted Courant bracket. In a similar manner, we twisted the C-bracket with the θ-transformation, obtained from the generator in double theory expressed in the non-canonical basis, in which Hamiltonian is diagonal, expressed in terms of T-dual metric. The θ-twisted C-bracket was obtained in the algebra of this generator. The bracket has exactly the same form as the B-twisted C-bracket, which was not the case for their analogous twisted Courant brackets. When we neglected all the T-dual coordinates in the expression for the θ-twisted C-bracket, the θ-twisted Courant bracket was obtained, while when we neglected all the initial coordinates, the B-twisted Courant bracket was obtained. We showed that in both twisted C-brackets, the isomorphism between mutually T-dual Courant algebroids is naturally included. We obtained the results regarding the twisted C-brackets and their derivations in [4].

The explanation of T-duality in terms of generalized geometry is still a work in progress, and there is a lot more work to be done. For instance, there are solved cases of equations of motions on background fields that from a simple geometric theory produce the T-dual theory that is not local. It would be important to see how the understanding of T-duality as the Courant algebroid isomorphism would generalize to such cases. The non-locality of the T-dual theories poses a challenge to understanding and interpreting the symmetries of their conformal field theory.

Additionally, there are challenges in the description of open string T-duality in terms of generalized geometry apparatus that were not touched upon in this dissertation. The open string action has to be extended with the terms related to the boundary conditions, which also change the symmetry generator. There was some work in literature with the aim to interpret the D-branes as Dirac structures [83, 84]. Hopefully, our results related to Dirac structures of various Courant algebroids and the fluxes on them might find the purpose in the challenges related to the open strings.

In the end, the description of Nature in terms of strings is contingent on the formulation of M-theory that, on one hand, gives effective action that describes gravity, while on the other hand, connects to a myriad of realizations of supersymmetric string theories. The fact that many of the superstring theories are connected by T-duality makes understanding it a priority. Therefore, further work will have to include the supersymmetry and see if isomorphism between Courant algebroids is still a valid description of T-duality.

Part VI

Appendix

Appendix A

Poisson manifolds

Let $\mathcal{M}$ be a manifold, and $C^\infty(\mathcal{M})$ the vector space of real valued functions on $\mathcal{M}$. A Poisson bracket on $\mathcal{M}$ is a map $\{,\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that satisfies:

1. Skew-symmetry: $\{f, g\} = -\{g, f\}$;
2. Leibniz rule: $\{f, gh\} = \{f, g\}h + \{f, h\}g$;
3. Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$

The Poisson bracket can be defined with the bi-vector θ by

$$\{f, g\} = \theta(df, dg)\,, \tag{A.1}$$

if the bi-vector satisfies the condition $[\theta, \theta]_S = 0$ (5.10). Two definitions are equivalent. The condition (5.10) ensures that the Jacobi identity is satisfied. The structure $(\mathcal{M}, \theta)$ is then called the Poisson manifold.

Appendix B

O(D, D) group

Indefinite orthogonal group $O(D,D)$ [85, 86] is defined as the Lie group of all linear transformations $\mathcal{O}$ of a $2D$-dimensional real vector space that leave invariant a non-degenerate symmetric bilinear form of signature (D,D)

$$\langle \mathcal{O}\Lambda_1, \mathcal{O}\Lambda_2\rangle = \langle \Lambda_1, \Lambda_2\rangle \,. \tag{B.1}$$

Let us express the general form of an $O(D,D)$ transformation as

$$\mathcal{O} = \begin{pmatrix} P^\mu_{\ \nu} & Q^{\mu\nu} \\ R_{\mu\nu} & S_\mu^{\ \nu} \end{pmatrix} \,, \tag{B.2}$$

where $P,\ Q,\ R,\ S$ are $D\times D$ matrices. Substituting (B.2) into (6.3), we obtain the constraint on these matrices:

$$P^T R + R^T P = 0\,, \quad P^T S + R^T Q = I\,, \quad Q^T S + S^T Q = 0\,, \tag{B.3}$$

where by I we denoted the $D\times D$ identity matrix.

From relation (6.3) we easily obtain that the inverse of the matrix $\mathcal{O}$ is given by

$$\mathcal{O}^{-1} = \eta^{-1}\mathcal{O}^T\eta\,, \tag{B.4}$$

or

$$\mathcal{O}^{-1} = \begin{pmatrix} S^T & Q^T \\ R^T & P^T \end{pmatrix} \,. \tag{B.5}$$

From the requirement $\mathcal{O}\mathcal{O}^{-1} = I$, we obtain another set of conditions on $P,\ Q,\ R,\ S$

$$PQ^T + QP^T = 0\,, \quad PS^T + QR^T = I\,, \quad RS^T + SR^T = 0\,. \tag{B.6}$$

The generators of $O(D,D)$ group include the following elements:

$$\mathcal{O}_A = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^T \end{pmatrix} , \tag{B.7}$$

$$\mathcal{O}_B = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix} , \tag{B.8}$$

and

$$\mathcal{O}_{\pm i} = \begin{pmatrix} 1 - E_i & \pm E_i \\ \pm E_i & 1 - E_i \end{pmatrix} , \tag{B.9}$$

where $(E_i)_{jk} = \delta^i_j \delta^i_k$. All other elements can be obtained from these generators.

Appendix C

Standard Courant algebroid

In this Appendix, we provide the proof for a claim that the structure $\Big(T\mathcal{M} \oplus T^\star\mathcal{M}, \langle,\rangle, [,]_\mathcal{C}, \pi\Big)$ consisting of the generalized tangent bundle over a smooth manifold, the natural inner product (6.2), and the Courant bracket is the Courant algebroid. Firstly, let us obtain the Courant algebroid differential operator (6.21), which we mark by

$$(\mathcal{D}f)^M = \begin{pmatrix} (\mathcal{D}^{(0)}f)^\mu \\ (\mathcal{D}^{(0)}f)_\mu \end{pmatrix} . \tag{C.1}$$

The right hand side of (6.21) becomes

$$\mathcal{L}_{\pi(\Lambda)} f = i_\xi \, df = \xi^\mu \partial_\mu f \, . \tag{C.2}$$

The left-hand side of (6.21) becomes

$$\langle \Lambda, \mathcal{D}^{(0)} f\rangle = \xi^\mu (\mathcal{D}^{(0)} f)_\mu + (\mathcal{D}^{(0)} f)^\mu \lambda_\mu \, . \tag{C.3}$$

Equating (C.2) and (C.3), we obtain

$$\mathcal{D}^{(0)} f = \begin{pmatrix} 0 \\ df \end{pmatrix} . \tag{C.4}$$

The differential operator $\mathcal{D}^{(0)}$ is basically just the exterior derivative d, but we chose the above notation so that its action on function gives generalized vector explicitly.

The first property (6.22) is evident when we act with the projection π to the definition of the Courant bracket (6.12), obtaining

$$\pi[\Lambda_1, \Lambda_2]_\mathcal{C} = [\xi_1, \xi_2]_L = [\pi(\Lambda_1), \pi(\Lambda_2)]_L \, . \tag{C.5}$$

To prove the second property, it is convenient to separate the vector and 1-form part of the left hand side of (6.23). The vector part becomes

$$[\xi_1, f\xi_2]_L = f[\xi_1, \xi_2]_L + (\mathcal{L}_{\xi_1} f)\, \xi_2 \,, \tag{C.6}$$

which is just the Leibniz rule for the Lie bracket (4.6). The 1-form part gives

$$\begin{aligned}
&\mathcal{L}_{\xi_1}(f\lambda_2) - \mathcal{L}_{(f\xi_2)}\lambda_1 - \frac{1}{2}d(i_{\xi_1}(f\lambda_2) - i_{(f\xi_2)}\lambda_1) = \\
&f\mathcal{L}_{\xi_1}\lambda_2 + (\mathcal{L}_{\xi_1} f)\, \lambda_2 - f\mathcal{L}_{\xi_2}\lambda_1 - df i_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}(f\lambda_2) - i_{(f\xi_2)}\lambda_1) = \\
&f\Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) + (\mathcal{L}_{\xi_1} f)\, \lambda_2 - \frac{1}{2}\langle\Lambda_1, \Lambda_2\rangle df \,,
\end{aligned} \tag{C.7}$$

where in the second line we applied (4.6) to the term $\mathcal{L}_{\xi_1}(f\lambda_2)$, and (4.18) to the term $\mathcal{L}_{(f\xi_2)}\lambda_1$. In the last line the Leibniz property for exterior derivative d was used. Combining relations (C.6) and (C.7) we obtain

$$[\Lambda_1, f\Lambda_2]_\mathcal{C} = f[\Lambda_1, \Lambda_2]_\mathcal{C} + (\mathcal{L}_{\pi(\Lambda_1)} f)\, \Lambda_2 - \frac{1}{2}\langle\Lambda_1, \Lambda_2\rangle \mathcal{D}^{(0)} f \,, \tag{C.8}$$

and therefore the second property (6.23) is satisfied.

For the third property (6.23) , we start from the first term on the right-hand side of it and write

$$\begin{aligned}
\langle[\Lambda_1, \Lambda_2] + \frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1, \Lambda_2\rangle, \Lambda_3\rangle &= \langle[\xi_1, \xi_2]_L \oplus (\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 + di_{\xi_2}\lambda_1), \xi_3 \oplus \lambda_3\rangle \\
&= i_{[\xi_1,\xi_2]_L}\lambda_3 + i_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - i_{\xi_2} d\lambda_1) \\
&= \mathcal{L}_{\xi_1} i_{\xi_2}\lambda_3 - i_{\xi_2}\mathcal{L}_{\xi_1}\lambda_3 + i_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - i_{\xi_2} d\lambda_1) \,,
\end{aligned} \tag{C.9}$$

where we firstly used the definition of the inner product (6.2) and Courant bracket (6.12), and afterwards the identity (4.20). Because we are working with a symmetric inner product, the second term of the right-hand side of (6.24) can be obtained from the previous relations by swapping $2 \leftrightarrow 3$

$$\langle\Lambda_2, [\Lambda_1, \Lambda_3] + \frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1, \Lambda_3\rangle\rangle = \mathcal{L}_{\xi_1} i_{\xi_3}\lambda_2 - i_{\xi_3}\mathcal{L}_{\xi_1}\lambda_2 + i_{\xi_2}(\mathcal{L}_{\xi_1}\lambda_3 - i_{\xi_3} d\lambda_1) \,. \tag{C.10}$$

Adding (C.9) and (C.10), we obtain

$$\begin{aligned}
&\langle[\Lambda_1, \Lambda_2] + \frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1, \Lambda_2\rangle, \Lambda_3\rangle + \langle\Lambda_2, [\Lambda_1, \Lambda_3] + \frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1, \Lambda_3\rangle\rangle \\
&= \mathcal{L}_{\xi_1}(i_{\xi_2}\lambda_3 + i_{\xi_3}\lambda_2) - (i_{\xi_3} i_{\xi_2} + i_{\xi_2} i_{\xi_3}) d\lambda_1 \\
&= \mathcal{L}_{\pi(\Lambda_1)}\langle\Lambda_2, \Lambda_3\rangle \,.
\end{aligned} \tag{C.11}$$

Here, we used (6.14) and (6.2), as well as (4.19). The third condition (6.24) has therefore been proven.

The fourth property (6.25) is evident from the fact that the inner product (6.2) of pure 1-forms is zero, i.e.

$$\langle \mathcal{D}^{(0)} f, \mathcal{D}^{(0)} g\rangle = \langle 0 \oplus df, 0 \oplus dg\rangle = 0\,. \tag{C.12}$$

The Jacobiator (6.19) for the Courant bracket can be easily obtained from definition of the Courant bracket. Firstly, we start with

$$\begin{aligned} \text{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) &= \xi \oplus \lambda \\ \xi &= [[\xi_1,\xi_2]_L,\xi_3]_L + cyclic = 0 \\ \lambda &= \mathcal{L}_{[\xi_1,\xi_2]_L}\lambda_3 - \mathcal{L}_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)) \\ &\quad -\frac{1}{2}d(i_{[\xi_1,\xi_2]_L}\lambda_3 - i_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1))) + cyclic\,. \end{aligned} \tag{C.13}$$

The vector part is just the Jacobi identity for the Lie bracket (4.7), and is therefore zero. The 1-form part is complicated, and it requires some more work, in order to be transformed properly. We use the Cartan formula (4.17) and nilpotency of the exterior derivative, as well as (4.20) to write

$$\begin{aligned} \lambda &= \frac{1}{2}\Big(\mathcal{L}_{[\xi_1,\xi_2]_L}\lambda_3 - \mathcal{L}_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)) \\ &\quad + i_{[\xi_1,\xi_2]_L}d\lambda_3 - i_{\xi_3}d(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1))\Big) + cyclic \\ &= \frac{1}{2}\Big(\mathcal{L}_{[\xi_1,\xi_2]_L}\lambda_3 - \mathcal{L}_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1) + \frac{1}{2}di_{\xi_3}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)) \\ &\quad + (\mathcal{L}_{\xi_1}i_{\xi_2} - i_{\xi_2}\mathcal{L}_{\xi_1})d\lambda_3 - i_{\xi_3}d(i_{\xi_1}d\lambda_2 - i_{\xi_2}d\lambda_1)\Big) + cyclic \\ &= \frac{1}{2}\Big(\mathcal{L}_{[\xi_1,\xi_2]_L}\lambda_3 - \mathcal{L}_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1) + \frac{1}{2}di_{\xi_3}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)) \\ &\quad + (i_{\xi_1}di_{\xi_2} + di_{\xi_1}i_{\xi_2} - i_{\xi_2}di_{\xi_1})d\lambda_3 - i_{\xi_3}d(i_{\xi_1}d\lambda_2 - i_{\xi_2}d\lambda_1)\Big) + cyclic\,. \end{aligned} \tag{C.14}$$

Firstly, using the definition of the Lie bracket (4.4), we conclude that

$$\begin{aligned} &\mathcal{L}_{[\xi_1,\xi_2]_L}\lambda_3 - \mathcal{L}_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1) + cyclic = \\ &(\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2} - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1})\lambda_3 + (\mathcal{L}_{\xi_2}\mathcal{L}_{\xi_3} - \mathcal{L}_{\xi_3}\mathcal{L}_{\xi_2})\lambda_1 + (\mathcal{L}_{\xi_3}\mathcal{L}_{\xi_1} - \mathcal{L}_{\xi_1}\mathcal{L}_{\xi_3})\lambda_2 \\ &-\mathcal{L}_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1) - \mathcal{L}_{\xi_1}(\mathcal{L}_{\xi_2}\lambda_3 - \mathcal{L}_{\xi_3}\lambda_2) - \mathcal{L}_{\xi_2}(\mathcal{L}_{\xi_3}\lambda_1 - \mathcal{L}_{\xi_1}\lambda_3) = 0\,. \end{aligned} \tag{C.15}$$

Secondly, we obtain

$$\begin{aligned} &(i_{\xi_1}di_{\xi_2} - i_{\xi_2}di_{\xi_1})d\lambda_3 - i_{\xi_3}d(i_{\xi_1}d\lambda_2 - i_{\xi_2}d\lambda_1) + cyclic = \\ &i_{\xi_1}di_{\xi_2}d\lambda_3 + i_{\xi_2}di_{\xi_3}d\lambda_1 + i_{\xi_3}di_{\xi_1}d\lambda_2 - i_{\xi_2}di_{\xi_1}d\lambda_3 - i_{\xi_3}di_{\xi_2}d\lambda_1 - i_{\xi_1}di_{\xi_3}d\lambda_2 \\ &-i_{\xi_3}di_{\xi_1}d\lambda_2 - i_{\xi_1}di_{\xi_2}d\lambda_3 - i_{\xi_2}di_{\xi_3}d\lambda_1 + i_{\xi_3}di_{\xi_2}d\lambda_1 + i_{\xi_1}di_{\xi_3}d\lambda_2 + i_{\xi_2}di_{\xi_1}d\lambda_3 = 0\,. \end{aligned} \tag{C.16}$$

Thirdly, we have

$$\begin{aligned}
&di_{\xi_3}(di_{\xi_1}\lambda_2 - di_{\xi_2}\lambda_1) + cyclic = \\
&di_{\xi_3}di_{\xi_1}\lambda_2 + di_{\xi_1}di_{\xi_2}\lambda_3 + di_{\xi_2}di_{\xi_3}\lambda_1 - di_{\xi_3}di_{\xi_2}\lambda_1 - di_{\xi_1}di_{\xi_3}\lambda_2 - di_{\xi_2}di_{\xi_1}\lambda_3 = \\
&(di_{\xi_2}di_{\xi_3} - di_{\xi_3}di_{\xi_2})\lambda_1 + (di_{\xi_3}di_{\xi_1} - di_{\xi_1}di_{\xi_3})\lambda_2 + (di_{\xi_1}di_{\xi_2} - di_{\xi_2}di_{\xi_1})\lambda_3\,,
\end{aligned} \tag{C.17}$$

and lastly

$$di_{\xi_1}i_{\xi_2}d\lambda_3 + cyclic = di_{\xi_1}i_{\xi_2}d\lambda_3 + di_{\xi_2}i_{\xi_3}d\lambda_1 + di_{\xi_3}i_{\xi_1}d\lambda_2\,. \tag{C.18}$$

Substituting (C.15), (C.16), (C.17) and (C.18) into (C.14), we obtain the Jacobiator of the Courant bracket

$$\begin{aligned}
\text{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) &= \frac{1}{2}\Big(di_{\xi_1}i_{\xi_2}d\lambda_3 + di_{\xi_2}i_{\xi_3}d\lambda_1 + di_{\xi_3}i_{\xi_1}d\lambda_2\Big) \\
&+\frac{1}{4}\Big((di_{\xi_2}di_{\xi_3} - di_{\xi_3}di_{\xi_2})\lambda_1 + (di_{\xi_3}di_{\xi_1} - di_{\xi_1}di_{\xi_3})\lambda_2 \\
&\qquad +(di_{\xi_1}di_{\xi_2} - di_{\xi_2}di_{\xi_1})\lambda_3\Big)\,.
\end{aligned} \tag{C.19}$$

Now in order to obtain the Nijenhuis operator, we substitute (6.12) and (6.2) in (6.20), and note that

$$\begin{aligned}
\text{Nij}(\Lambda_1,\Lambda_2,\Lambda_3) &= \frac{1}{6}\langle[\Lambda_1,\Lambda_2]_C,\Lambda_3\rangle + cyclic\,, \\
\langle[\Lambda_1,\Lambda_2]_C,\Lambda_3\rangle &= i_{[\xi_1,\xi_2]_L}\lambda_3 + i_{\xi_3}\Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big)\,.
\end{aligned} \tag{C.20}$$

If we take the exterior derivative of the above relation, using (4.17) and (4.20), we obtain

$$\begin{aligned}
d\langle[\Lambda_1,\Lambda_2]_C,\Lambda_3\rangle &= di_{[\xi_1,\xi_2]_L}\lambda_3 + di_{\xi_3}\Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) \\
&= di_{\xi_1}di_{\xi_2}\lambda_3 - di_{\xi_2}di_{\xi_1}\lambda_3 - di_{\xi_2}i_{\xi_1}d\lambda_3 + di_{\xi_3}i_{\xi_1}d\lambda_2 - di_{\xi_3}i_{\xi_2}d\lambda_1 \\
&\quad +\frac{1}{2}(di_{\xi_3}di_{\xi_1}\lambda_2 - di_{\xi_3}di_{\xi_2}\lambda_1)\,.
\end{aligned} \tag{C.21}$$

Again, we can easily add the cyclic permutations od terms that have similar form. For instance, we have

$$\begin{aligned}
&di_{\xi_1}di_{\xi_2}\lambda_3 - di_{\xi_2}di_{\xi_1}\lambda_3 + \frac{1}{2}(di_{\xi_3}di_{\xi_1}\lambda_2 - di_{\xi_3}di_{\xi_2}\lambda_1) + cyclic = \\
&di_{\xi_1}di_{\xi_2}\lambda_3 - di_{\xi_2}di_{\xi_1}\lambda_3 + di_{\xi_2}di_{\xi_3}\lambda_1 - di_{\xi_3}di_{\xi_2}\lambda_1 + di_{\xi_3}di_{\xi_1}\lambda_2 - di_{\xi_1}di_{\xi_3}\lambda_2 \\
&+\frac{1}{2}\Big(di_{\xi_3}di_{\xi_1}\lambda_2 - di_{\xi_3}di_{\xi_2}\lambda_1 + di_{\xi_1}di_{\xi_2}\lambda_3 - di_{\xi_1}di_{\xi_3}\lambda_2 + di_{\xi_2}di_{\xi_3}\lambda_1 - di_{\xi_2}di_{\xi_1}\lambda_3\Big) = \\
&\frac{3}{2}\Big((di_{\xi_2}di_{\xi_3} - di_{\xi_3}di_{\xi_2})\lambda_1 + (di_{\xi_3}di_{\xi_1} - di_{\xi_1}di_{\xi_3})\lambda_2 + (di_{\xi_1}di_{\xi_2} - di_{\xi_2}di_{\xi_1})\lambda_3\Big)\,.
\end{aligned} \tag{C.22}$$

The remaining terms from (C.21) become

$$
\begin{aligned}
&-di_{\xi_2}i_{\xi_1}d\lambda_3 + di_{\xi_3}i_{\xi_1}d\lambda_2 - di_{\xi_3}i_{\xi_2}d\lambda_1 + cyclic = \\
&-di_{\xi_2}i_{\xi_1}d\lambda_3 + di_{\xi_3}i_{\xi_1}d\lambda_2 - di_{\xi_3}i_{\xi_2}d\lambda_1 - di_{\xi_3}i_{\xi_2}d\lambda_1 + di_{\xi_1}i_{\xi_2}d\lambda_3 - di_{\xi_1}i_{\xi_3}d\lambda_2 \\
&-di_{\xi_1}i_{\xi_3}d\lambda_2 + di_{\xi_2}i_{\xi_3}d\lambda_1 - di_{\xi_2}i_{\xi_1}d\lambda_3 = \\
&3\Big(di_{\xi_1}i_{\xi_2}d\lambda_3 + di_{\xi_2}i_{\xi_3}d\lambda_1 + di_{\xi_3}i_{\xi_1}d\lambda_2\Big)\,,
\end{aligned} \tag{C.23}
$$

where we used the interior product property (4.19). The derivative $\mathcal{D}^{(0)}$of the Nijenhuis operator for the Courant bracket is obtained by substituting (C.22) and (C.23) in (C.21)

$$
\begin{aligned}
\mathcal{D}^{(0)}\text{Nij}(\Lambda_1,\Lambda_2,\Lambda_3) \;=\; & \frac{1}{2}\Big(di_{\xi_1}i_{\xi_2}d\lambda_3 + di_{\xi_2}i_{\xi_3}d\lambda_1 + di_{\xi_3}i_{\xi_1}d\lambda_2\Big) \\
& +\frac{1}{4}\Big((di_{\xi_2}di_{\xi_3} - di_{\xi_3}di_{\xi_2})\lambda_1 + (di_{\xi_3}di_{\xi_1} - di_{\xi_1}di_{\xi_3})\lambda_2 \\
& \quad +(di_{\xi_1}di_{\xi_2} - di_{\xi_2}di_{\xi_1})\lambda_3\Big)\,.
\end{aligned} \tag{C.24}
$$

Comparing relations (C.19) and (C.24), we finally prove the last Courant algebroid compatibility condition (6.26)

$$
\text{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) = \mathcal{D}^{(0)}\text{Nij}(\Lambda_1,\Lambda_2,\Lambda_3)\,. \tag{C.25}
$$

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Curriculum vitae

Ilija Ivanišević was born on July 17, 1991, in Mostar. He completed his secondary education at the Mathematical High School in 2010. In 2014, he graduated from the Faculty of Physics with an average grade of 9.66/10. The following year, he obtained his Master's degree at the same faculty with a thesis entitled "T-dualization in curved space" under the supervision of Dr. Ljubica Davidović. During his studies, he was a scholarship recipient of the City of Belgrade (2007-2010), the Ministry of Education, Science and Technological Development of the Republic of Serbia (2010-2013, 2015-2016), and the Fund for Young Talents (2014-2015).

He enrolled in doctoral studies in the field of Quantum Fields, Particles, and Gravity in 2015, and passed his doctoral exams with an average grade of 9/10. Since 2018, he has been employed at the Institute of Physics, where he has been working under the mentorship of Dr. Ljubica Davidović. His area of interest is string theory, specifically T-duality and its relation to generalized geometry. He has published four peer-reviewed papers in top-tier international journals whose contents provide the basis for the present thesis. Ilija presented his work in student talks at the 10th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics in 2019 in Belgrade, and at the 17th DIAS-TH Winter School "Supersymmetry and Integrability" organized by the Bogoliubov Laboratory of Theoretical Physics, JINR (Dubna, Russia).

During his studies, Ilija represented the University of Belgrade in international university debate competitions, achieving numerous successes, winning over ten competitions, and advancing to the finals of the European University Debate Championship in Tallinn in 2017.

Изјава о ауторству

Име и презиме аутора **Илија Иванишевић**

Број индекса **8016/2015**

Изјављујем

да је докторска дисертација под насловом

Courant algebroids in bosonic string theory

(Курантови алгеброиди у бозонској теорији струна)

- резултат сопственог истраживачког рада;
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Име и презиме аутора **Илија Иванишевић**

Број индекса **8016/2015**

Студијски програм **квантна поља, честице и гравитација**

Наслов рада **Courant algebroids in bosonic string theory** (**Курантови алгеброиди у бозонској теорији струна)**

Ментор **др. Љубица Давидовић**

Изјављујем да је штампана верзија мог докторског рада истоветна електронској верзији коју сам предао/ла ради похрањена у **Дигиталном репозиторијуму Универзитета у Београду.**

Дозвољавам да се објаве моји лични подаци везани за добијање академског назива доктора наука, као што су име и презиме, година и место рођења и датум одбране рада.

Ови лични подаци могу се објавити на мрежним страницама дигиталне библиотеке, у електронском каталогу и у публикацијама Универзитета у Београду.

Потпис аутора

У Београду, **21.04.2023**

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Изјава о коришћењу

Овлашћујем Универзитетску библиотеку „Светозар Марковић“ да у Дигитални репозиторијум Универзитета у Београду унесе моју докторску дисертацију под насловом:

Courant algebroids in bosonic string theory

(Курантови алгеброиди у бозонској теорији струна)

која је моје ауторско дело.

Дисертацију са свим прилозима предао/ла сам у електронском формату погодном за трајно архивирање.

Моју докторску дисертацију похрањену у Дигиталном репозиторијуму Универзитета у Београду и доступну у отвореном приступу могу да користе сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons) за коју сам се одлучио/ла.

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