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EKSTREMNE VREDNOSTI U NIZOVIMA NEZAVISNIH SLUČAJNIH VELIČINA SA MEŠAVINAMA RASPODELA

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EXTREME VALUES IN SEQUENCES OF INDEPENDENT RANDOM VARIABLES WITH MIXED DISTRIBUTIONS

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Abstract

This thesis has been written under the supervision of my mentor Prof. Dr. Pavle Mladenović at the University of Belgrade, Faculty of Mathematics in the academic year 2012-2013. The title of this thesis is "Extreme values in sequences of independent random variables with mixed distributions". For good survey of the field, see Resnick, S. I. [25] and Samorodnitsky, Taqqu [27]. The thesis is divided into two chapters. Chapter 1 is divided into 7 sections. In this chapter, we focus on classical results in extreme value theory. We discuss maxima and minima in the first section, univariate extreme value theory in the second section, max-stable distributions in the third section, peaks over threshold models in the fourth section, domain of attraction of the extremal type distributions in the fifth section, tails in the sixth section and tail equivalence in the seventh section.

Chapter 2 is divided into 9 sections. In this chapter, we discuss mixed distributions in the first section, mixture of normal distributions in the second section, mixture of Cauchy distributions in the third section, stable distributions in the fourth section, properties of stable random variables in the fifth section, infinitely divisible distributions in the sixth section. Sections 7 and 8 contain the main results for this dissertation. Mixtures of stable distributions are described in the seventh section and mixtures of an infinite sequence of independent normally distributed variables in the eighth section. Conclusion and future research are in the ninth section.

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EHFAYED SHNEINA

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INTRODUCTION

Extreme Value distributions arise as limiting distributions for maximum or minimum (extreme values) of a sample of independent and identically distributed random variables, as the sample size increases. Extreme Value Theory (EVT) is the theory of modeling and measuring events which occur with very small probability. This implies its usefulness in risk modeling as risky events per definition happen with low probability. Thus, these distributions are important in Statistics and Probability. These models, along with the Generalized Extreme Value Distribution (GEVD), are widely used in risk management, finance, insurance, economics, hydrology, material sciences, telecommunications, and many other fields dealing with extreme events. There are three fundamental mathematical results that illustrate the importance of extreme value theory (EVT) in risk management applications:

1- Extremal types Theorem.
2- Domain of attraction Theorem.
3- Standard for choosing a high Threshold.

The main results of this thesis are contained in papers Shneina, E. K. [6]; Shneina, E. K. and Božin, V. [7]. These papers extend the results of Mladenović, P., [20]. Namely, we determined the type of extreme value distributions and the corresponding normalizing constants for sequences of independent identically distributed (i.i.d.) random variables which are mixtures of stable distributions [6]. Also, we showed that a common distribution function of extreme value of a mixture of an infinite sequence of independent identically distributed (i.i.d.) normally distributed random variables, and also determined the normalizing constants [7].

Chapter 1

Classical Results in Extreme Value Theory

1.1 Maxima and Minima

(i) **Maxima:** Let $X_1, X_2, \ldots X_n$ be a sequence of independent identically distributed (i.i.d.) random variables with distribution function $F(x)$ and put

$$M_n = \max\{X_1, X_2, \ldots, X_n\} = \max_{1\le i\le n} X_i,$$

$$m_n = \min\{X_1, X_2, \ldots, X_n\} = \min_{1\le i\le n} X_i.$$

The relation between max and min is:

$$\min\{X_1, X_2, \ldots, X_n\} = -\max\{-X_1, -X_2, \ldots, -X_n\},$$
$$\min_{1\le i\le n} X_i = -\max_{1\le i\le n}(-X_i).$$

Then the distribution function of M_n is

$$P\{M_n \le x\} = P\{X_1 \le x, X_2 \le x, \ldots, X_n \le x\} = P\{X_1 \le x\} \ldots P\{X_n \le x\},$$
$$P\{M_n \le x\} = [F(x)]^n = F^n(x),\ n \ge 1\ \ x \epsilon \mathcal{R}, n \epsilon N.$$

(ii) **Minima:** Generally, the results for minima can be deduced from the corresponding results for maxima by writing $\min_{1\le i\le n} X_i = -\max_{1\le i\le n}(-X_i)$. In conjunction with minima, it can be useful to present results in terms of the survival function $\bar{F} = 1 - F$.

We have $P\{m_n > x\} = (1 - F(x))^n = (\bar{F}(x))^n = \bar{F}^n(x)$.

Therefore, the distribution function of the minima is

$$P\{m_n \le x\} = 1 - [1 - F(x)]^n = 1 - \bar{F}^n(x).$$

The class of Extreme Value Distributions (EVD's) essentially involves three types of extreme value distributions, types I, II and III, defined below.

Definition 1.1.1. (Extreme Value Distributions for maxima).
The following are *the standard extreme value distribution functions for maxima*:

1. Gumbel (type I): $(x) = G_0(x) = \exp(-e^{-x})\ , \quad - \quad < x < + \quad ;$

2. Fréchet (type II): $_{\alpha}(x) = G_{1,\alpha}(x) = \begin{cases} \exp(-x^{-\alpha}), & if\ x > 0, \\ 0, & if\ x \le 0, \end{cases} \quad for\ some\ \alpha > 0;$

3. Weibull (type III): $_{\alpha}(x) = G_{2,\alpha}(x) = \begin{cases} \exp(-(-x)^{\alpha}), & if\ x \le 0, \\ 1, & if\ x > 0. \end{cases} \quad for\ some\ \alpha > 0.$

Definition 1.1.2. (Extreme Value Distributions for minima).

The standard extreme value distributions for minima are defined as:

$$ ^*(x) = 1 - \ (-x), \qquad ^*(x) = 1 - \ _\alpha(-x) \ \ and \ \ ^*(x) = 1 - \ _\alpha(-x). $$

Then the following are *the standard extreme value distribution functions for minima*:

1. Gumbel (type I)**:** $^*(x) = G_0^*(x) = 1 - \exp(-e^x), \quad - \ < x < + \ ;$

2. Fréchet (type II)**:** $^*_\alpha(x) = G_{1,\alpha}^*(x) = \begin{cases} 1 - \exp(-(-x)^{-\alpha}), & if\ x < 0, \\ 1, & if\ \ x \geq 0, \end{cases} \quad for\ some\ \alpha > 0;$

3. Weibull (type III): $^*_\alpha(x) = G_{2,\alpha}^*(x) = \begin{cases} 1 - \exp(-x^\alpha), & if\ x \geq 0, \\ 0, & if\ x < 0. \end{cases} \quad for\ some\ \alpha > 0.$

1.2. Univariate Extreme Value Theory

Extreme Value Theory (EVT) is a branch of probability theory and statistics which deals with large values in a data set. It has become more widespread in the past decade as a tool for risk management in different areas. The theory can be used by banks to estimate extreme investment losses, enables insurance companies to price their products and aids the government to budget for possible storms, earthquakes and other natural disasters. Generally, there are two approaches to studying the distribution of extreme values namely, Block Maxima Models (M_n) and Peaks Over Threshold Models (POT). In the univariate these approaches respectively lead to the Generalized Extreme Value Distributions (GEVD) (Fisher-Tippett, 1928 [11]; Gnedenko, 1943 [12]) with location μ, scale σ and shape ξparameters and the Generalized Pareto Distributions (GPD) (Pickands, 1975 [22]; Davison and Smith, 1990 [5]) with shape ξ and scale σ parameters.

1.2.1. Block Maxima ($\boldsymbol{M_n}$) Models

Suppose $X_1, X_2, \ldots$ are independent identically distributed (i.i.d) random variables with common distribution function *F*. Let $M_n = \max\{X_1, X_2, \ldots, X_n\} = \max_{1 \leq i \leq n} X_i$, denote the maximum of the first n random variables and let $x_F = sup\{x : F(x) < 1\}$ denote the right endpoint of *F*. We have $P\{M_n \leq x\} = P\{X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x\} = F^n(x),\ n \geq 1\ \ x\epsilon\mathcal{R}, n\epsilon N.$
M_n Converges almost surely to x_F whether it is finite or infinite. The limit theory in univariate extremes seeks norming constants $a_n > 0,\ b_n$ and a nondegenerate *G* such that the distribution function of a normalized version of M_n converges to *G*, i.e.

$$ \lim_{n\to} P\left(\frac{M_n - b_n}{a_n} \leq x\right) = \lim_{n\to} F^n(a_n x + b_n) = G(x), \qquad (1.1) $$

If this holds for suitable choices of a_n *and* b_n then we say that G is an extreme value distribution function and F is in the domain of attraction of G, written as $F \epsilon DA(G)$.
We say further that two distribution functions G and G^* are of *the same type*,
if $G^*(x) = G(ax + b)$ for some $a > 0$, b and all x.

The Extremal Types Theorem (Fisher and Tippett, 1928 [11]; Gnedenko, 1943 [12]; de Haan, 1970 [18]) characterizes the limit distribution function G as of the type of one of the following three classes:

1. Gumbel (type I): $(x) = G_0(x) = \exp(-e^{-x})$, $- \quad < x < + \quad ;$

2. Fréchet (type II): $_{\alpha}(x) = G_{1,\alpha}(x) = \begin{cases} \exp(-x^{-\alpha}), & if\ x > 0, \\ 0, & if\ x \leq 0, \end{cases}$ $\quad for\ some\ \alpha > 0;$

3. Weibull (type III): $_{\alpha}(x) = G_{2,\alpha}(x) = \begin{cases} \exp(-(-x)^{\alpha}), & if\ x \leq 0, \\ 1, & if\ x > 0. \end{cases}$ $\quad for\ some\ \alpha > 0.$

Thus, any extreme value distribution can be classified as one of Type I, II or III.
The three types are often called *the Gumbel, Fréchet and Weibull* types, respectively.

1.2.2. Generalized Extreme value distribution (GEVD)

The role of the generalized extreme value (GEV) distributions in the Theory of Extremes is analogous to that of the normal distribution in central limit theory for sums of random variables. Assume $X_1, X_2, \ldots$ are independent identically distributed (i.i.d.) with finite variance and writing $S_n = X_1 + X_2 + \cdots + X_n$ for the sum of the first (n) random variables, the standard version of central limit theorem (*CLT*) says that appropriately normalized sums $\frac{S_n - a_n}{b_n}$ converge in distribution to the standard normal distribution as (n) goes to infinity. The appropriate normalization used sequences of normalizing constants (a_n) and (b_n) defined by $a_n = nE(X_1)$ and $b_n = \sqrt{n\, var(X_1)}$.
In mathematical notation we have $\lim_{n\to} \ P[b_n^{-1}(S_n - a_n) \leq x] = \quad (x),\ x \epsilon \mathcal{R}$.
For more details see Coles, Stuart (2001) [4]; Embrechts, Klüppelberg and Mikosch (1997) [8]; Leadbetter, Lindgren and Rootzen (1983) [17] and Resnick (1987) [25].

Definition 1.2.3. (The Generalized Extreme Value (GEV) distribution)
The classical extreme value theory is based on three asymptotic extreme value distributions identified by Fisher and Tippett (1928) [11]. And the distribution function of the (GEV) given by

$$F_{GEV}(x;\xi,\sigma,\mu) = \begin{cases} exp\left[-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{\frac{-1}{\xi}}\right]; & \xi \neq 0 \\ exp\left(-e^{-\left(\frac{x-\mu}{\sigma}\right)}\right); \; \xi = 0; \end{cases}$$

where $1+\xi\left(\frac{x-\mu}{\sigma}\right) > 0$, $\mu\epsilon\mathcal{R}$ is the location parameter, $\sigma > 0$ the scale parameter, and $\xi\epsilon\mathcal{R}$ the shape parameter.
The parameter ξ is known as the shape parameter of the GEV distribution and $F_{GEV}\,(x;\xi,\sigma,\mu)$ defines a type of distribution:
If $\xi > 0$ then $F_{GEV}(x;\xi,\sigma,\mu)$ is Fréchet distribution.
If $\xi = 0$ then $F_{GEV}(x;0,\sigma,\mu)$ is Gumbel distribution.
If $\xi < 0$ then $F_{GEV}(x;\xi,\sigma,\mu)$ is Weibull distribution.

All the graphics below are made in XTREMES program [28].

The following figure (1.1) contains distribution functions that belong to the class of the generalized extreme value distributions (Fréchet distribution is represented with red line, Gumbel distribution with black line and Weibull distribution with green line).

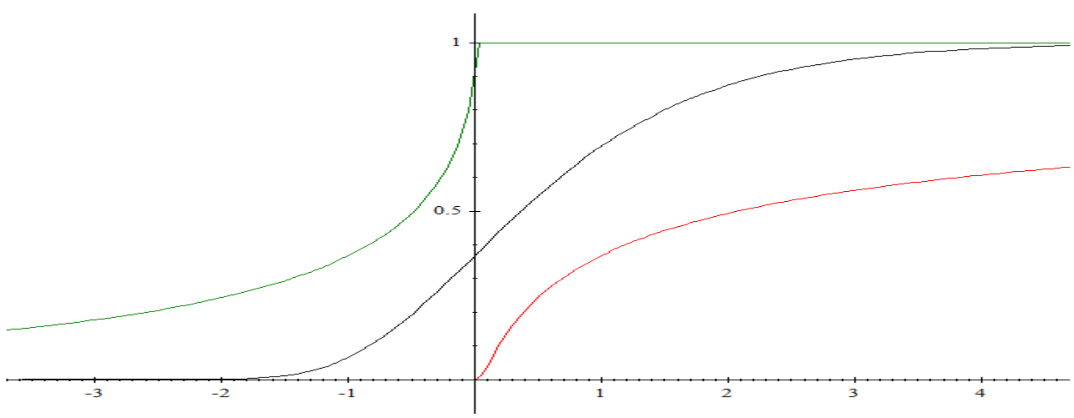

Figure (1.1): *Generalized extreme value distributions*.

It has the following density function

$$f_{gev}(x;\xi,\sigma,\mu) = \begin{cases} \frac{1}{\sigma}\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-\left(1+\frac{1}{\xi}\right)} \quad exp\left[-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{\frac{-1}{\xi}}\right]; & \xi \neq 0 \\ \frac{1}{\sigma}\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right\}; & \xi = 0, \end{cases}$$

for $1+\xi\left(\frac{x-\mu}{\sigma}\right) > 0.$

The following figure (1.2) contains the probability density functions of the generalized extreme value distributions (Fréchet distribution is represented with red line, Gumbel distribution with black line and Weibull distribution with green line).

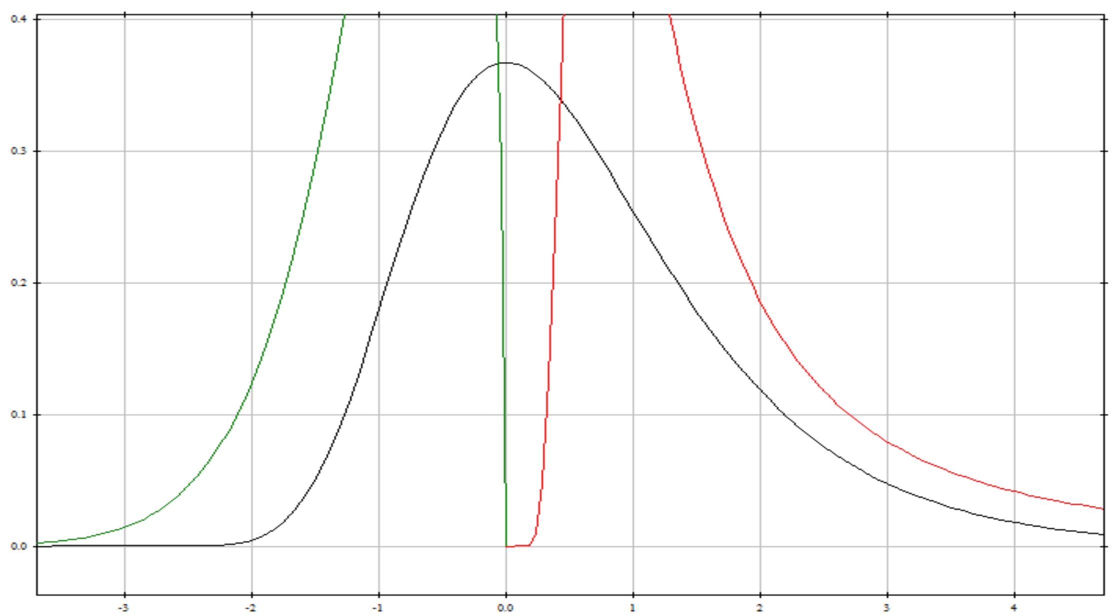

Figure (1.2): *Probability density functions of the generalized extreme value distributions*.

Remark: Note the differences in the ranges of interest for the three extreme value distributions: Gumbel is unlimited, Fréchet has a lower limit, while the Weibull has an upper limit.

The figure (1.3) below shows Densities for Gumbel, Fréchet and Weibull distribution functions respectively from left to right.

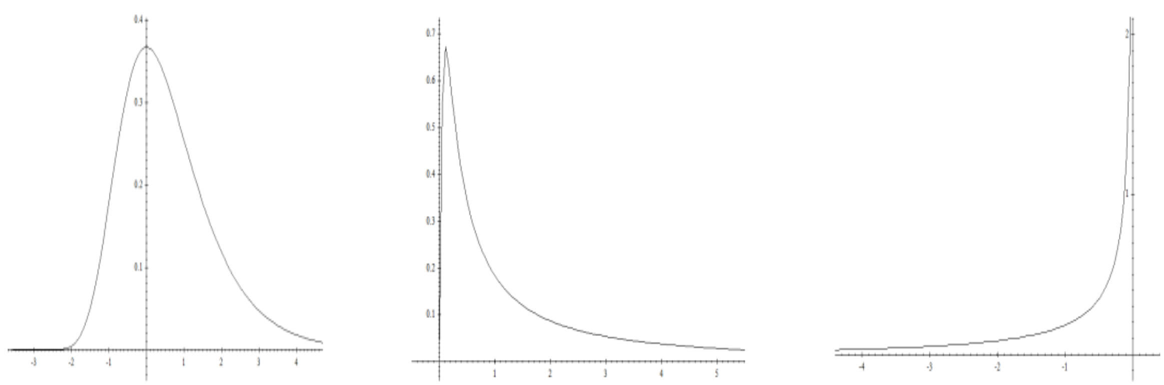

Figure (1.3): *For Gumbel, Fréchet and Weibull densities respectively from left to right.*

1.2.4. Extremal Types Theorem

The extreme type theorems play a central role of the study of extreme value theory.
In the literature, Fisher and Tippett (1928) [11], were the first who discovered the extreme type theorems and later these results were proved in complete generality by Gnedenko (1943) [12]. Leadbetter, Lindgren and Rootzen (1983) [17] and Resnick (1987) [25], are excellent reference books on the probabilistic aspect.
A recent book by Embrechts, Klüppelberg, and Mikosch (1997) [8], gives an excellent viewpoint of modelling extremal events. The extreme type theorems say that for a sequence of independent identically distributed (i.i.d.) random variables with suitable normalizing constants, the limiting distribution of maximum statistics, if it exists, follows one of three types of extreme value distributions.

The fundamental extreme value theorem (Fisher and Tippett, (1928) [11]; Gnedenko, (1943) [12]) ascertains the Generalized Extreme Value distribution in the von Mises parametrization (von Mises, (1936) [26]) as an unified version of all possible non-degenerate weak limits of partial maxima of sequences comprising independent identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$. That is:

Theorem 1.2.5. (Fisher and Tippett, (1928) [11]; Gnedenko, (1943) [12])
Let $\{X_n, n \geq 1\}$ *be a sequence of independent identically distributed (i.i.d.) random variables with distribution F, and suppose there exist normalizing constants* $a_n > 0,\ b_n \epsilon \mathcal{R},\ n \geq 1$ *such that*

$$P[a_n^{-1}(M_n - b_n) \leq x] = F^n(a_n x + b_n) \to G(x), \tag{1.2}$$

where $G(x)$ *is a non-degenerate limiting distribution. Then* $G(x)$ *belongs to the type of one of the following three distributions:*

1. Gumbel (type I): $\Lambda(x) = G_0(x) = exp(-e^{-x})\ , \quad -\infty < x < +\infty\,;$

2. Fréchet (type II): $\Phi_\alpha(x) = G_{1,\alpha}(x) = \begin{cases} exp(-x^{-\alpha}), & if\ x > 0, \\ 0, & if\ x \leq 0, \end{cases} \quad for\ some\ \alpha > 0;$

3. Weibull (type III): $\Psi_\alpha(x) = G_{2,\alpha}(x) = \begin{cases} exp(-(-x)^{\alpha}), & if\ x \leq 0, \\ 1, & if\ x > 0. \end{cases} \quad for\ some\ \alpha > 0;$

where α *is a positive constant*. We refer to $= G_0 = EVo$, $_\alpha = G_{1,\alpha} = EV_1$ and $_\alpha = G_{2,\alpha} = EV_2$ as the extreme value distributions, while the constants $\boldsymbol{a_n}$ and b_n from (1.2) are called *the normalizing constants*.

The details of the proof can be found in Resnick (1987) [25], Proposition 0.3, pp. 9-11.

1.3. Max-stable distributions

Definition 1.3.1. A non-degenerate distribution function *G* is called *Max-stable* if there exist real constants $a_n > 0,\ b_n \epsilon \mathcal{R},\ n \geq 1$ such that $for\ all\ x, \quad G^n(a_n x + b_n) = G(x).$

Then $G(x)$ is one of the following three types: Type I: Gumbel, Type II: Fréchet and Type III: Weibull. The next result (Theorem (1.4.1) in Leadbetter et al. (1983) [17]) shows this property.

Theorem 1.3.2. *Every max-stable distribution is of extreme value type, i.e. equal to* $G(ax + b)\ for\ some\ a > 0, b$. *Conversely, each distribution of extreme value type is max-stable.*

Definition 1.3.3. G is *strictly stable* iff $b_n = 0$, for all n.
We give a list of normalizing constants when the max-stable distribution function is one of the standard extreme value (EV) distribution functions:

(i) Gumbel: $\Lambda = G_0$: $a_n = 1,\ b_n = \ln n$.

(ii) Fréchet: $\Phi_\alpha = G_{1,\alpha}$: $a_n = n^{\frac{1}{\alpha}},\ \ b_n = 0$.

(iii) Weibull: $\Psi_\alpha = \ G_{2,\alpha}$: $a_n = n^{\frac{-1}{\alpha}},\ b_n = 0.$

Examples 1.3.4:

Example (1): $G_0(x) = exp(-e^{-x}),\ a_n = 1,\ b_n = \ln n\,,\ \ x\epsilon\mathcal{R}$.

Then $\left(G_0(a_n x + b_n\ )\right)^n = \left(G_0(x + \ln n\ \ )\right)^n = \left(\exp\left(-e^{-(x+\ln n)}\right)\right)^n$

$=exp(-e^{-x}).\,{}^{1}\!/_{n}\,.\,n = exp(-e^{-x}) = \ G_0(x).$

Example (2): $G_{1,\alpha}(x) = \exp(-x^{-\alpha})\,, a_n = n^{\frac{1}{\alpha}},\ \ b_n = 0.$

Then $\left(G_{1,\alpha}(a_n x + b_n\ )\right)^n = \left(G_{1,\alpha}\left(n^{\frac{1}{\alpha}}x\ \right)\right)^n = \left(\mathrm{e}^{-\left(n^{\frac{1}{\alpha}}x\right)^{-}}\right)^n = e^{(-x)^{-\alpha}.\frac{1}{n}.n} = exp(-(x))^{-\alpha} =$

$G_{1,\alpha}(x).$

Example (3): $G_{2,\alpha}(x) = \exp(-(-x)^\alpha),\ \ a_n = n^{\frac{-1}{\alpha}}, b_n = 0$. Then

$\left(G_{2,\alpha}(a_n x + b_n\ )\right)^n = \left(G_{2,\alpha}\left(n^{\frac{-1}{\alpha}}x\ \right)\right)^n = \left(\mathrm{e}^{-\left(-n^{\frac{-1}{\alpha}}x\right)}\right)^n = e^{(-(-x))^{\alpha}.\frac{1}{n}.n} \ = \ \ exp(-(-x))^\alpha =$

$G_{2,\alpha}(x).$

The class of slowly and regularly varying functions are introduced by Karamata, J. (1930) [15].

Definition1.3.5. (Slowly Varying and Regularly Varying Function)

A positive measurable function L on $(0,\ )$ is called

(i) *Slowly varying at infinity* (write $L\epsilon\mathcal{R}V_0$) if $\lim_{t\to} \frac{L(tx)}{L(t)} = 1,\ x > 0.$

(ii) *Regularly varying at infinity with index* $\rho\big(write\ L\epsilon\mathcal{R}V_\rho\big)if$ $\lim_{t\to} \frac{L(tx)}{L(t)} = x^\rho\ , x > 0.$

Further information can be found in Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987) [3]; Embrechts and Mikosch (1997) [8]; de Haan (1970) [18]; Resnick (1987) [25] and many other textbooks.

Examples 1.3.6:

Example (1): If $L(t) = \ln t\ \ then\ \lim_{t\to} \frac{L(tx)}{L(t)} = \lim_{t\to} \frac{\ln(tx)}{\ln(t)} = \lim_{t\to} \frac{\ln t + \ln x}{\ln t} = 1.$

Satisfies (i) in definition (1.3.5) then $L(t) = \ln t$ is slowly varying at infinity.

Example (2): If $L(t) = t^\rho \ln x\ \ then\ \lim_{t\to} \frac{L(tx)}{L(t)} = \lim_{t\to} \frac{(tx)\ \ln(tx)}{t\ \ln(t)} = x^\rho.$

Thus $L(t)$ is a regularly varying at with index ρ, and satisfies (ii) in definition (1.3.5).

Example (3): The functions: $e^x,\ \sin(x+2)\ and\ \ \exp\{\log x\}$, are not regularly varying.

Example (4): The functions: $\log x,\ \log(1 + \mathrm{x})\ ,\ \log\log(e + x)\ and\ \exp\{(\log x)^\alpha\}$,

$0 < \alpha < 1,$ are slowly varying.

1.4. Peaks Over Threshold (POT) Models

1.4.1. Generalized Pareto Distribution (GPD)

The role of Generalized Pareto Distribution (GPD) in Extreme Value Theory (EVT) is as a natural model for the excess distribution over a high threshold. And the distribution function of the (GPD) is given by

$$F_{GPD}(x;\sigma,\xi) = \begin{cases} 1 - \left(1 + \frac{\xi x}{\sigma}\right)^{\frac{-1}{\xi}} & ;\ \xi \neq 0, \\ 1 - exp\left(\frac{-x}{\sigma}\right); & \xi = 0, \end{cases}$$

where $\sigma > 0, and\ x \geq 0\ when\ \xi \geq 0,$
and $0 \leq x \leq \frac{-\sigma}{\xi}\ \ when\ \ \xi < 0.$

The parameters ξ and σ are respectively, as the shape and scale parameters.
For more information about (GPD) see Embrechts, Klüppelberg and Mikosch (1997) [8]; Davison and Smith (1990) [5] and many other textbooks.

The following figure (1.4) contains distribution functions that belong to the class of the generalized pareto distributions (Pareto distribution is represented with red line, Exponential distribution with black line and Beta distribution with green line).

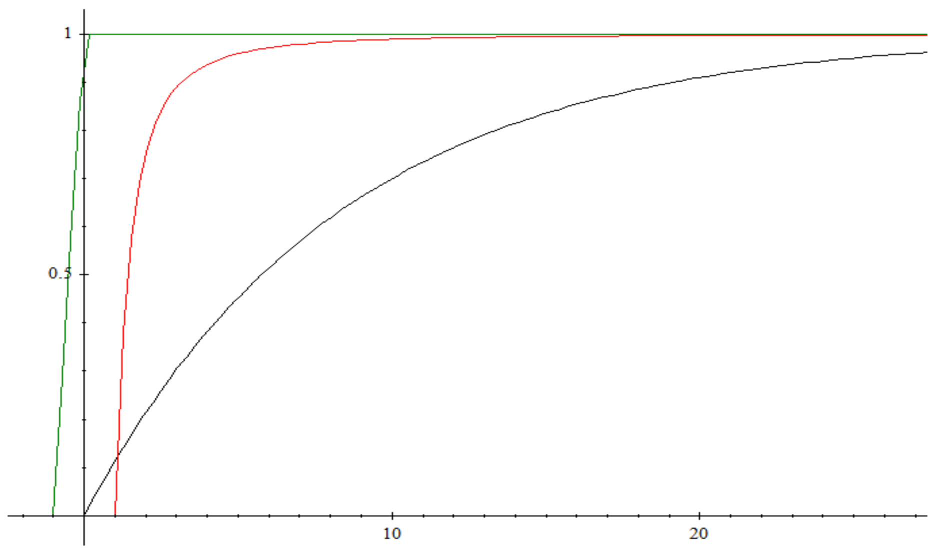

Figure (1.4): *Generalized Pareto distribution functions.*

The following analytical relationship exists between the Generalized Pareto Distribution (GPD) and the generalized extreme value (GEV) distribution functions $F_{GEV}(x)$ for ξ-parameterization:
$F_{GPD}(x) = 1 + \ln(F_{GEV}(x))$, where $\alpha = \frac{1}{\xi} = \xi^{-1}$, $if \ln(F_{GEV}(x)) > -1$.
The three limiting distributions in the GPD family include the Pareto, Beta, and Standard exponential distribution functions:

(i) Exponential (GP_0): $F_{GP_0}(x) = \begin{cases} 1 - e^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

(ii) Pareto (GP_1), $\alpha > 0$: $F_{GP_{1,\alpha}}(x) = \begin{cases} 1 - x^{-\alpha}, & x \geq 1, \\ 0, & x < 1. \end{cases}$

(iii) Beta (GP_2), $\alpha < 0$: $F_{GP_{2,\alpha}}(x) = \begin{cases} 1 - (-x)^{-\alpha}, & -1 \leq x \leq 0, \\ 0, & x < -1. \end{cases}$

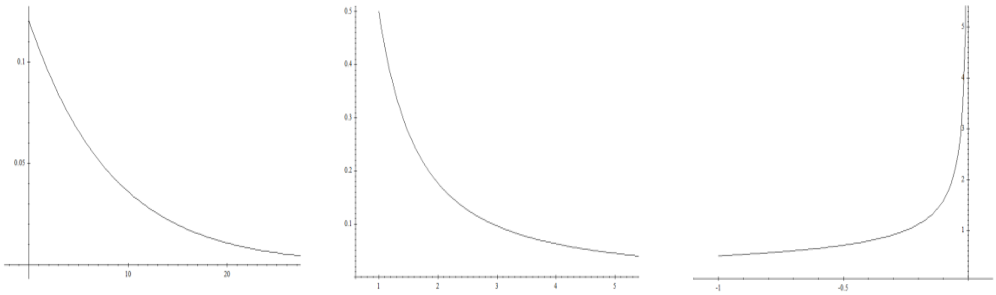

Figure (1.5): *Densities for Exponential, Pareto and Beta distributions respectively from left to right.*

1.4.2. Relationship between Extreme Value (EV) and Generalized Pareto (GP) distributions

The relationship for three different GP distributions are given as follows:

(i) The exponential distribution function corresponds to the Gumbel distribution as follows:

$$F_{GP_0}(x) = 1 + \ln(F_{EV_0}(x)) = 1 - \exp(-x),\ x \geq 0.$$

(ii) The Pareto(or ordinary Pareto) distribution function corresponds to the Fréchet distribution as follows:

$$F_{GP_{1,\alpha}}(x) = 1 + \ln(F_{EV_{1,\alpha}}(x)) = 1 - x^{-\alpha},\quad for\ x \geq 1, \alpha > 0\ .$$

(iii) The Beta distribution function corresponds to the Weibull distribution as follows:

$$F_{GP_{2,\alpha}}(x) = 1 + \ln(F_{EV_{2,\alpha}}(x)) = 1 - (-x)^{-\alpha},\quad for\ \ -1 \leq x \leq 0, \qquad \alpha < 0.$$

1.4.3. Modeling Excess distributions

Let X be a random variable with distribution function F. The distribution of excesses over a threshold *u* has distribution function:

$$F^{(u)}(x) = P(X - u \leq x | X > u) = \frac{F(x+u) - F(u)}{1 - F(u)}, \qquad for\ 0 \leq x < x_F - u,$$

where $x_F = \sup\{x: F(x) < 1\} \leq \infty$, $\quad x_F =$ right endpoint of F.

The mean excess function of a random variable X with finite mean is given by

$$e_{F^u} = E(X - u | X > u) = \frac{1}{1-F(u)} \int_u^{\infty} \big(1 - F(x)\big) dx,\ \ u > 0. \tag{1.3}$$

The mean excess function e_{F^u} expresses the mean of $F^{(u)}$ as a function of u.
In survival analysis, the mean excess function is known as the mean residual life function and gives the expected residual lifetime for components with different ages.

Remark: The distribution function $F^{(u)}$ is called *the conditional excess distribution function*.

Examples 1.4.4: (Excess distribution of exponential and GPD)

(1) If F is the distribution function of an exponential random variable then $F^{(u)}(x) = F(x);\ \ for\ all\ x.$

Proof: Since $F(x) = \begin{cases} 1-e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$ where $\lambda > 0$,

$$\Rightarrow F^{(u)}(x) = \frac{F(x+u)-F(u)}{1-F(u)} = e^{-\lambda u}\,\frac{\left[1-e^{-\lambda x}\right]}{e^{-\lambda u}} = 1-e^{-\lambda x} = F(x),$$

$$\Rightarrow F^{(u)}(x) = F(x).$$

(2) If X has distribution function
$F\ = F_{GPD}(x;\sigma,\xi) \Rightarrow F^{(u)}(x) = F_{GPD}(x;\sigma(u),\xi),\quad \sigma(u) = \sigma + \xi(u),$
where $0 \leq x <\quad if\ \xi > 0\ and\ 0 \leq x \leq \frac{-\sigma}{\xi}\ if\ \xi < 0.$

Proof: $F^{(u)}(x) = \frac{F(x+u)-F(u)}{1-F(u)}$ $,\Rightarrow F^{(u)}(x) = \frac{F_{GPD}(x+u)-F_{GPD}(u)}{1-F_{GPD}(u)} =$

$$\frac{\left(1-\left(1+\frac{\xi(x+u)}{\sigma}\right)^{-\xi^{-1}}\right)-\left(1-\left(1+\frac{\xi u}{\sigma}\right)^{-\xi^{-1}}\right)}{1-\left(1-\left(1+\frac{\xi u}{\sigma}\right)^{-\xi^{-1}}\right)} =$$

$$\frac{\left(1+\frac{\xi u}{\sigma}\right)^{-\xi^{-1}}-\left(1+\frac{\xi(x+u)}{\sigma}\right)^{-\xi^{-1}}}{\left(1+\frac{\xi u}{\sigma}\right)^{-\xi^{-1}}} = 1-\left(\frac{\sigma+\xi(x+u)}{\sigma+\xi u}\right)^{-\xi^{-1}} =$$

$$= 1-\left(\frac{\sigma+\xi x+\sigma(u)-\sigma}{\sigma+\sigma(u)-\sigma}\right)^{-\xi^{-1}} = 1-\left(\frac{\sigma(u)+\xi x}{\sigma(u)}\right)^{-\xi^{-1}} = 1-\left(1+\frac{\xi x}{\sigma(u)}\right)^{-\xi^{-1}}$$

$$= F_{GPD}(x;\sigma(u),\xi),\ \ where\ \ \sigma(u) = \sigma+\xi(u) \Rightarrow \xi(u) = \sigma(u)-\sigma.$$

(3) Then mean excess function of the GPD can be calculated by $E(X) = \frac{\sigma}{1-\xi}$, and by formula (1.3),

$$e_{F^u} = \frac{\sigma(u)}{1-\xi} = \frac{\sigma+\xi u}{1-\xi}, \qquad \sigma+\xi u > 0,$$

where $0 \le u <$ $\quad if\ 0 \le \ \xi < 1\ and\ 0 \le u < \frac{-\sigma}{\xi}\ if\ \xi < 0.$

The Pickands-Balkema-de Haan limit Theorem (Balkema and de Haan, (1974) [1]; Pickands, (1975) [22]) is a key result in Extreme Value Theory (EVT) and explains the importance of the Generalized Pareto Distribution (GPD).

Theorem 1.4.5. (Pickands-Balkema-de Haan Theorem (1974)).
Suppose that $X_1, X_2, \dots, X_n$ are n independent realizations of a random variable X with a distribution function $F(x)$. Let x_F be the finite or infinite right endpoint of the distribution F. The distribution function of the excesses over certain high threshold u is given by

$$F^{(u)}(x) = P(X-u \le x | X > u) = \frac{F(x+u)-F(u)}{1-F(u)}, \ \ for\ 0 \le x < x_F - u.$$

If $F \epsilon\ DA\big(F_{GEV}(x;\xi,\sigma,\mu)\big)$, then there exists a positive measurable function $\sigma(u)$ such that

$$\lim_{u \to x_F} sup\left|F^{(u)}(x) - F_{GPD}(x;\sigma(u),\xi)\right| = 0.$$

1.5. Domain of attraction of the extremal type distributions

Definition 1.5.1. (Domain of Attraction (DA)). Suppose $\{X_n,\ n \ge 1\}$ is a sequence of independent identically distributed (i.i.d.) random variables with the common distribution function *F*. The distribution *F* belongs to *the domain of attraction* of the extreme value distribution *G*, $F\epsilon DA(G)$ if there exist constants $a_n > 0,\ b_n \epsilon \mathcal{R},\ n \ge 1$ such that

$F^n\,(a_n x + b_n) = P[M_n \le a_n x + b_n] \to G(x)$, as $n \to$, where $M_n = \max_{1 \le i \le n} X_i$.

Leadbetter et al. (1983) [17] gave a comprehensive account of necessary and sufficient conditions for $F\epsilon DA(G)$ and characterizations of $a_n\ and\ b_n$ when *G* is one of the three extreme value distribution functions above.

The following theorem from Leadbetter, M.R., Lindgreen, G. and Rootzén, H. (1983) [17] is very useful in finding the domain of attraction of *F,* and gives necessary and sufficient conditions:

Theorem 1.5.2. *The following conditions are necessary and sufficient for a distribution function F to belong to the domain of attraction of the three extremal types:*

Gumbel (type I): *There exists a strictly positive function $g(t)$ defined on the set $(-\infty, x_F)$, such that for every real number x the equality $lim_{t\uparrow x_F} \frac{1-F(t+xg(t))}{1-F(t)} = e^{-x}$ holds true, where*

$g(t) = \frac{\int_x^{x_F}(1-F(t))dt}{(1-F(x))}$, *for $x < x_F$.*

Fréchet (type II): $x_F = +\infty$, *and* $lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}$, *for some* $\alpha > 0$ *and all* $x > 0$.

Weibull (type III): $x_F < +\infty$, *and* $lim_{h\downarrow 0} \frac{1-F(x_F-hx)}{1-F(x_F-h)} = x^{\alpha}$, *for some* $\alpha > 0$ *and all* $x > 0$.

The proofs of the theorem can be found in Leadbetter et al. (1983) [17]; Resnick (1987) [25]; etc.

Theorem 1.5.3. (Characterization of DA (G))

The distribution function F belongs to the domain of attraction of the extreme value distribution G with norming constants $a_n > 0$, $b_n \epsilon \mathcal{R}$ iff

$lim_{n\to\infty} n\bar{F}(a_n x + b_n) = -\ln G(x)$, $x \epsilon \mathcal{R}$, *when* $G(x) = 0$ *the limit is interpreted as infinity,*

where $\bar{F}(a_n x + b_n) = 1 - F(a_n x + b_n)$.

For more information see Embrechts and Mikosch (1997) [8]; Resnick (1987) [25].

For every standard extreme value distribution one can characterize its domain of attraction. Using the concept of regular variation this is not too difficult for the Fréchet distribution $_{\alpha}$ and the weibull distribution $_{\alpha}$. The domain of attraction of the Gumbel distribution is not so easily characterized; it consists of distribution functions whose right tail decreases to zero faster than any power function.

Definition 1.5.4. (Von Mises function). Let *F* be a distribution function with right endpoint $x_F \le \infty$. Suppose there exists some $z < x_F$ such that *F* has representation $\bar{F}(x) = c.exp\left\{-\int_z^x \frac{1}{a(t)} dt\right\}$, $z < x < x_F$, where *c* is some positive constant, *a*(.) is a positive and absolutely continuous function with density a' *and* $\lim_{x\uparrow x_F} a'(x) = 0$.

Then *F* is called a *Von Mises function*, the function *a*(.) is the auxiliary function of *F*. For more details see Resnick (1987) [25], proposition 1.4 and de Haan (1972) [2].

Theorem 1.5.5. (Von Mises Condition).

(i) *Let F be an absolutely continuous distribution function with density f satisfying* $lim_{x\to\infty} \frac{xf(x)}{\bar{F}(x)} = \alpha > 0,\ \ then\ \ F\epsilon DA(\Phi_\alpha)$.

(ii) *Let F be an absolutely continuous distribution function with density f which is positive on some finite interval (z,* x_F*). If*
$lim_{x\uparrow x_F} \frac{(x_F - x)f(x)}{\bar{F}(x)} = \alpha > 0,\ \ then\ \ F\epsilon DA(\Psi_\alpha)$.

For more details see Resnick (1987) [25], proposition 1.15 and proposition 1.16, pp. 63.

Properties of Von Mises functions 1.5.6. Every Von Mises function *F* is *absolutely continuous* on (z, x_F) with positive density *f*. The auxiliary function can be chosen as $a(x) = \frac{\bar{F}(x)}{f(x)}$. Moreover, the following properties hold:

(i) If $x_F = \infty$, then $\bar{F}\ \epsilon RV_{-\infty}$ and $\lim_{x\to\infty} \frac{xf(x)}{\bar{F}(x)} = \infty$.

(ii) If $x_F < \infty$, then $\bar{F}(x_F - x^{-1})\epsilon RV_{-\infty}$ and $\lim_{x\uparrow x_F} \frac{(x_F - x)f(x)}{\bar{F}(x)} = \infty$.

For more details see Embrechts and Mikosch (1997) [8], pp.140.

We give some examples of Von Mises functions. See Embrechts and Mikosch (1997) [8], pp. 139.

Example (1): (Exponential distribution)

$\bar{F}(x) = e^{-\lambda x}, x \geq 0, \lambda > 0$. *F* is a Von Mises function with auxiliary function $a(x) = \lambda^{-1}$.
Proof: $\bar{F}(x) = 1 - F(x) = e^{-\lambda x}$, $F(x) = 1 - e^{-\lambda x}$, $F'(x) = f(x) = \lambda e^{-\lambda x}$,

then the auxiliary function $a(x) = \frac{\bar{F}(x)}{f(x)} = \frac{e^{-\lambda x}}{\lambda e^{-\lambda x}} = \frac{1}{\lambda} = \lambda^{-1}$.

Example (2): (Weibull distribution)

$\bar{F}(x) = e^{-cx^{\tau}}, x \geq 0,\ \ c, \tau > 0.$ F is a Von Mises function with auxiliary function

$$a(x) = c^{-1}\tau^{-1}x^{1-\tau}, x > 0.$$

Proof: $\bar{F}(x) = 1 - F(x) = e^{-cx^{\tau}},\quad F(x) = 1 - \ e^{-cx^{\tau}}, F'(x) = f(x) = e^{-cx^{\tau}}(c\tau x^{\tau-1}),$

then the auxiliary function $a(x) = \frac{\bar{F}(x)}{f(x)} = \frac{e^{-cx^{\tau}}}{e^{-cx^{\tau}}(c\tau x^{\tau-1})} = \frac{1}{c\tau x^{\tau-1}} = c^{-1}\tau^{-1}x^{1-\tau}, x > 0.$

Theorem 1.5.7. (Von Mises (1936)) [26]. *F is absolutely continuous distribution function and* $x_F = sup\{x: F(x) < 1\}$. *If*

- *(i)* $F''(x) < 0,\ for\ all\ x\epsilon(z, x_F),\ x_F \leq \infty.$
- *(ii)* $F'(x) = 0,\ for\ x \geq x_F.$
- (iii) $lim_{x\to x_F}\frac{F'(x)(1-F(x))}{\left(F'(x)\right)^2} = 1, \qquad then\ F\epsilon DA(\Lambda)$.

This is sufficient conditions for continuous function.

For more details see Resnick (1987) [25]; Embrechts and Mikosch (1997) [8].

Example: Let $F(x) = 1 - e^{-x},\ x > 0.$

$$Then\ F'(x) = f(x) = e^{-x}, \qquad x \geq 0,$$

$and\ f(x) = \frac{F'(x)(1-F(x))}{\left(F'(x)\right)^2} = \frac{1-F(x)}{F'(x)} = \frac{e^{-x}}{e^{-x}} = 1,$

Therefore $f'(x) = 0$, and $F\epsilon DA(\ \ )$. See Resnick (1987) [25], pp. 42.

The following theorems can be found in Leadbetter et al. (1983) [17]; Balkema, and de Haan (1972) [2]; and Resnick (1987) [25], are giving necessary and sufficient conditions:

Theorem 1.5.8. (Gnedenko (1943) [12]; De Haan (1970) [18]). *For a distribution function F set* $H(x) = \frac{1}{1-F(x)}$, $\quad x_F = sup\{t : F(t) < 1\}$, *so that* $H^{\leftarrow} = H^{-1}$ *is defined on* $(1, \infty)$. *The following are equivalent:*

- *(i)* $F\epsilon DA(\Lambda)$, *if there exist constants* $a_n > 0,\ b_n\epsilon\mathcal{R},\ n\epsilon N$ *such that*

  $lim_{n\to\infty}\ F^n\ (a_n x + b_n) = exp\ (-e^{-x}),$ *for all x.*

- *(ii)* $H\epsilon\Gamma$, *there exist g such that for every real number x:* $lim_{t\uparrow x_F}\frac{1-F(t+xg(t))}{1-F(t)} = e^{-x}$.

- *(iii)* $H^{\leftarrow} = H^{-1}\epsilon\Pi$, *there exist a such that for all* $x > 0$: $\quad lim_{t\to\infty}\frac{H^{-1}(tx)-H^{-1}(t)}{a(t)} = ln\,x.$

For more details see Resnick (1987) [25], proposition 0.10., pp. 28-30.

Theorem 1.5.9. (De Haan, (1970) [18]). $F \epsilon DA(\Lambda)$ *iff*

$$lim_{x \to x_{x_F}} \frac{(1-F(x)) \int_x^{x_F} \int_y^{x_F} (1-F(t))\, dt\, dy}{\left(\int_x^{x_F} (1-F(t)) dt\right)^2} = 1,$$

in this case $\frac{1}{1-F} \epsilon \Gamma$, *and the auxiliary function can be chosen* $g(t) = \frac{\int_x^{x_F} \int_y^{x_F} (1-F(t))\, dt\, dy}{\int_x^{x_F} (1-F(t)) dt}$, *or* $g(t) = \frac{\int_x^{x_F} (1-F(t)) dt}{(1-F(x))}$ *,and norming constants can be chosen*

$a_n = g(b_n),\ \ b_n = F^{-1}\left(1 - \frac{1}{n}\right).$

For more details see Resnick (1987) [25], proposition 1.9., pp. 48-50.

Theorem 1.5.10. (De Haan, (1970) [18]). $F \epsilon DA(\Lambda)$ *iff*

$lim_{x \to x_F} \frac{\int_x^{x_F} (1-F(t))^{\alpha} dt}{(1-F(x)) \int_x^{x_F} (1-F(t))^{\alpha-1} dt} = \frac{\alpha-1}{\alpha}, for\ some\ \alpha > 1$. In this case it's true for all $\alpha > 1$.

For more details see Resnick (1987) [25], proposition 1.10., pp. 50-52.

Remark 1.5.11. The corresponding characterizations of a_n *and* b_n are:

1. Gumbel (type I):
$a_n = g(b_n),\ \ b_n = F^{-1}\left(1 - \frac{1}{n}\right)$. Where the auxiliary function can be chosen
$g(t) = \frac{\int_x^{x_F} (1-F(t)) dt}{(1-F(x))}$.

For more details see Resnick (1987) [25], Corollary 1.7 and proposition 1.9., pp. 48-50.

2. Fréchet (type II):

$a_n = F^{-1}\left(1 - \frac{1}{n}\right),\ \ b_n = 0.$

For more details see Resnick (1987) [25], pp. 54-57.

3. Weibull (type III):
$a_n = x_F - F^{-1}\left(1 - \frac{1}{n}\right),\ b_n = x_F.$

For more details see Resnick (1987) [25], pp. 59-62. and Embrechts and Mikosch (1997) [8], pp. 135.

The domain of attraction of the distribution function F is determined by the asymptotic behavior of the tail $1-F(x)$, as $x \to +$. The following theorem from Leadbetter et al. (1983) [17] is important for determining of normalizing constants a_n *and* b_n in (1.2).

Theorem 1.5.12. *Let $\{X_n\}$ be an independent identically distributed (i.i.d.) sequence random variables. Let $\tau \in [0, +\ )$, and suppose that $\{u_n\}$ is a sequence of real numbers, such that*
$n(1-F(u_n)) \to \tau$, as $n \to$ (1.4)
then $P\{M_n \leq u_n\} \to e^{-\tau}$, as $n \to$. (1.5)
Conversely, if (1.5) holds for some τ, $[0, +\ )$, then (1.4) holds.

The proofs of the theorem can be found in Leadbetter et al. (1983) [17]; Resnick (1987) [25]; etc.

Some other theoretical results may be very useful for finding the DA (G) of F and finding the normalizing constants. Those results and examples whose distributions belong to each of the three domain of attraction can be found in Leadbetter et al. (1983) [17]; Resnick (1987) [25]; etc.

1.5.13. Examples of Domain of Attraction

Example (1): We consider now the Pareto distribution.
We consider $F(x) = 1 - x^{-\alpha}$, $\alpha > 0$, $x \geq 1$. We have $\lim_{t\to} \frac{1-F(tx)}{1-F(t)} = \frac{(tx)^{-\alpha}}{(t)^{-\alpha}} = x^{-\alpha}$.

In this example, the distribution function $F(x)$ belongs to the domain of attraction of the function $G_1(x)$, and we have the type (II) of extreme value distribution, i.e. there exist constants $a_n$and b_n, such that the following equality holds true:

$$P\left\{M_n \leq \frac{x}{a_n} + b_n\right\} \to \exp(-x^{-\alpha}).$$

We now determine the constants $a_n$and b_n.

Let us first determine the constant u_n, such that $1 - F(u_n) \sim \frac{1}{n} x^{-\alpha}$, , $\alpha > 0$, as $n \to$, *i.e.*

$1 - F(u_n) \sim {}^{\tau}/_{n}$, $\tau = x^{-\alpha}$, $\tau > 0$,

$(u_n)^{-\alpha} \sim \frac{x^{-\alpha}}{n}, as\ n \to \infty,$

$(u_n)^{-\alpha} \sim \left(\frac{x}{n^{\frac{-1}{\alpha}}}\right)^{-\alpha}, as\ n \to \infty,$ and we obtain

$u_n \sim n^{\frac{1}{}} x$, as $n \to \infty$.

Using Theorem **1.5.12** we obtain

$$P\left\{M_n \le n^{\frac{1}{\alpha}} x\right\} \to e^{-x^{-\alpha}}, \quad as\ n \to \infty,$$

we get $a_n = n^{\frac{-1}{\alpha}},\ b_n = 0\ and\ G(x) = e^{-x^{-\alpha}}$.

Example (2): If $F(x) = \begin{cases} 1 - e^{\frac{1}{x}}, & x < 0, \\ 1, & x \ge 0. \end{cases}$

Determine the type of extreme value distribution and the normalizing constants?

We consider $F(x) = 1 - e^{\frac{1}{x}}$.

We have $\frac{1-F(t+xg(t))}{1-F(t)} = \frac{e^{\frac{1}{t+xg(x)}}}{e^{\frac{1}{t}}} = e^{\frac{-xg(t)}{t(t+xg(t))}} = e^{\frac{-xt^2}{t^2+xt^3}} = e^{\frac{-x}{1+xt}}, \quad put\ g(t) = t^2, \Rightarrow$

$\lim_{t \to 0^-} e^{\frac{-x}{1+xt}} = e^{-x}, as\ t \to 0^-, \Rightarrow$

$$\lim_{t \to 0^-} \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x} \quad , as\ t \to 0^-,$$

then $F \epsilon DA($ (x)) (of type I, Gumbel distribution).

We now determine the constants a_n, b_n.

We consider $F(x) = 1 - e^{\frac{1}{x}},\ x < 0, \quad \Rightarrow 1 - F(x) = e^{\frac{1}{x}},\ x < 0,$ (1.6)

$$then\ 1 - F(u_n) \sim {}^{\tau}/_{n}, \quad \tau > 0, \quad \tau = e^{\frac{1}{x}}$$

$1 - F(u_n) \sim {}^{e^{\frac{1}{x}}}/_{n}, \quad x < 0,$ (note that $\tau = e^{\frac{1}{x}} \Rightarrow \ln \tau = \frac{1}{x}, \tau > 0$).

From (1.6) $\Rightarrow e^{\frac{1}{u_n}} \sim {}^{e^{\frac{1}{x}}}/_{n} \Rightarrow \ln e^{\frac{1}{u_n}} \sim \ln({}^{e^{\frac{1}{x}}}/_{n}) \Rightarrow \frac{1}{u_n} \sim \ln({}^{e^{\frac{1}{x}}}/_{n}) \Rightarrow u_n \sim \left(\ln {}^{e^{\frac{1}{x}}}/_{n}\right)^{-1},$

$$u_n \sim \left(\ln e^{\frac{1}{x}} - \ln n\right)^{-1} \Rightarrow u_n \sim \left(\frac{1}{x} - \ln n\right)^{-1} \Rightarrow u_n \sim (\ln \tau - \ln n)^{-1},$$

$P\{M_n \le u_n\} \to e^{-\tau}, \quad as\ n \to \quad ,$

$$\tau = e^{-x}, \quad x \epsilon R, \quad u_n = (\ln \tau - \ln n)^{-1},$$

$$u_n = (\ln e^{-x} - \ln n)^{-1} = (-x - \ln n)^{-1} = -(x + \ln n)^{-1},$$

$$\Rightarrow \; P\{M_n \leq -(x + \ln n)^{-1}\} \to \exp(-e^{-x}),$$

But $(x + \ln n)^{-1} = \left(\ln n\left(\frac{x}{\ln n} + 1\right)\right)^{-1} = (\ln n)^{-1}\left(1 - \frac{x}{\ln n} + \sigma\left(\frac{1}{\ln n}\right)\right) = \frac{1}{\ln n} - \frac{x}{(\ln n)^2} + \sigma\left(\frac{1}{(\ln n)^2}\right)$,

$$\Rightarrow \{M_n \leq -(x + \ln n)^{-1}\} = \left\{M_n \leq -\frac{1}{\ln n} + \frac{x}{(\ln n)^2} + \sigma\left(\frac{1}{(\ln n)^2}\right)\right\}$$
$$= \left\{M_n \leq \frac{-\ln n + x + \sigma(1)}{(\ln n)^2}\right\} = \{(\ln n)^2 . M_n + \ln n \leq x + \sigma(1)\},$$

$= \left\{(\ln n)^2 \left[M_n + \frac{1}{\ln n}\right] \leq x + \sigma(1)\right\} \Rightarrow M_n \leq \frac{x}{(\ln n)^2} - \frac{1}{\ln n}$,

Using Theorem **1.5.12** we obtain $P\left\{M_n \leq \frac{x}{(\ln n)^2} - \frac{1}{\ln n}\right\} \to e^{-e^{\frac{1}{x}}}$.

Thus $a_n = (\ln n)^2$ *and* $b_n = -\frac{1}{\ln n} = -(\ln n)^{-1}$.

Example (3): Suppose $X_1, X_2, \ldots$ be financial loss, independent identically distributed (i.i.d.) with distribution function F and defined as; $F(x) = 1 - \exp(-\lambda x)$ *where* $\lambda > 0, x > 0$. Choose normalizing sequences

$a_n = \frac{1}{\lambda}$, $b_n = \frac{\ln n}{\lambda}$, Calculate $F^n(a_n x + b_n)$?

Proof: Since $F(x) = 1 - \exp(-\lambda x)$, then $F^n(x) = (1 - \exp(-\lambda x))^n$

So that $F^n(a_n x + b_n) = F^n\left(\frac{1}{\lambda}x + \frac{\ln n}{\lambda}\right) = \left[1 - exp\left(-\lambda\left(\frac{1}{\lambda}x + \frac{\ln n}{\lambda}\right)\right)\right]^n =$

$$= [1 - \exp(-x - \ln n)]^n = [1 - \exp(-x) . \exp(\ln n^{-1})]^n =$$

$\left(1 - \frac{\exp(-x)}{n}\right)^n = \left(1 - \frac{1}{n}\exp(-x)\right)^n$, then $G(x) = \lim_{n\to\infty} F^n(a_n x + b_n) =$

$$= \lim_{n\to\infty}\left(1 - \frac{1}{n}\exp(-x)\right)^n = exp(-e^{-x}) = G_0(x) = \Lambda(x).$$

Thus $F \epsilon DA($ (x)).

1.6. Tails

The following definitions are from [14].

Definition 1.6.1. (Fat –tailed distribution)

The distribution of random variable X is said to have a *fat tail* if

$P[X > x] = \bar{F}(x) = 1 - F(x) \sim x^{-\alpha},\ as\ x \to \infty, \alpha > 0.$

Remark: Cauchy distributions are examples of fat-tail distributions.

Definition 1.6.2. (Heavy-tailed distribution)

The distribution of a random variable X with distribution function F is said to have a *heavy right tail* if $\lim_{x\to\infty} e^{\lambda x}\bar{F}(x) = \infty,\ for\ all\ \lambda > 0,\ \ \bar{F}(x) = 1 - F(x),\ \bar{F}(x\ ) = \ P[X > x].$

Definition 1.6.3. (Long-tailed distribution)

The distribution of a random variable X with distribution function F is said to have a *long right tail* if $\lim_{x\to\infty} P[X > x + t : X > x] = 1, for\ all\ t > 0$, or equivalently

$$\bar{F}(x + t) \sim \bar{F}(x), as\ x \to \infty.$$

1.7. Tail equivalence

The following Theorems and Results are from Feller, (1966) [10]; Gnedenko, (1943) [12]; Resnick, (1971) [24] and Resnick, (1987) [25].

Definition 1.7.1. (Tail equivalence)

Two distribution functions *F(x)* and *G(x)* are called *tail equivalent* if they have the same right endpoint, i.e. if $x_0^F = x_0^G = x_0$ and $\lim_{x \to x_0^F} \frac{1-F(x)}{1-G(x)} = A,\ for\ some\ A > 0\ and$

$x_0 = inf\{x: F(x) = 1\}$.

Definition 1.7.2. (Tail equivalence)

Two distribution functions *F(x)* and *G(x)* are *right tail equivalent* iff

$$x_0^F = x_0^G = x_0, \qquad 1 - F(x) \sim 1 - G(x)\ as\ x \to x_0-;\ and\ \lim_{x \to x_0-} \frac{1 - F(x)}{1 - G(x)} = 1.$$

Definition 1.7.3. Two distribution functions *U(x)* and *V(x)* are of *the same type* if for some

$A > 0, B \epsilon R,\quad V(x) = U(Ax + B),$ for all *x*.

Theorem 1.7.4. *Suppose U(x) and V(x) are two non-degenerate distribution functions. If for a sequence* $F_n(x)$ *of distribution functions there exist constants* $a_n > 0, b_n \epsilon R,\ n \geq 1\ and$ $\alpha_n > 0, \beta_n \epsilon R,$ such that $F_n(a_n\, x + b_n) \to^c U(x), \quad F_n(\alpha_n x + \beta_n) \to^c V(x),$

$$\Rightarrow \frac{\alpha_n}{a_n} \to A > 0, \quad \frac{\beta_n - b_n}{a_n} \to B \epsilon R\ \ and\ V(x) = U(Ax + B).$$

Remark 1.7.5. The set of normalizing constants $a_n > 0, b_n \epsilon R,\ n \geq 1$ is asymptotically equivalent to the set of normalizing constants $\alpha_n > 0,\ \beta_n \epsilon R, n \geq 1\ iff\ \frac{\alpha_n}{a_n} \to 1,\ \frac{\beta_n - b_n}{a_n} \to 0.$

Chapter 2

Extreme Values for Mixtures

2.1. Mixed distributions

Let X_1 and X_2 be random variables with distribution functions $F_1(x)$and $F_2(x)$, respectively, and

$$X = \begin{cases} X_1 & with\, probability\; p, \\ X_2 & with\, probability\; q, \end{cases} \quad \text{where } p + q = 1.$$

The distribution function of the random variable X is given by
$F(x) = P\{X \le x\} = pP\{X_1 \le x\} + qP\{X_2 \le x\} = pF_1(x) + qF_2(x),$

The distribution function F is called *the mixture of the distributions* determined by the functions $F_1\; and\; F_2$.

We shall consider some examples of sequences of independent random variables with common mixed distribution. In these cases we are going to determine the type of extreme value distribution and the normalizing constants.

2.2 Mixture of normal distributions

The following example can be found in [20], but without a proof.

Example (2.2.1):

Let (X_n) be a sequence of independent random variables with normal $\mathcal{N}(0,1)$ distribution and $M_n = \max\{X_1, \dots, X_n\}$. As is well known, the limiting distribution of the maximum M_n is given by $P\{a_n(M_n - b_n) \le x\} \to \exp(-x^{-\alpha})$, $n \to$,

where the normalizing constants a_n and b_n are given by

$$a_n = \sqrt{2 \ln n}, \quad b_n = \sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi}{2\sqrt{2 \ln n}}.$$

Proof: Let $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$. We shall use the following asymptotic relation

$$1 - \Phi(x) \sim x^{-1}\varphi(x), \quad x \to \infty. \tag{2.1}$$

If $X\epsilon\mathcal{N}(\mu,\sigma)$, then distribution function of the random variable X can be represented in the form $F(x) = P\{X \le x\} = \Phi\left(\frac{x-\mu}{\sigma}\right)$, using this representation of $F(x)$ we obtain

$$1 - F(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right) = 1 - \Phi(t),$$

$$1 - F(t) \sim \frac{1}{t}\varphi(t), \qquad t \to \infty,$$

$$1 - F(t) \sim \frac{1}{t}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}, \qquad t \to \infty;$$

$$\frac{1 - F(t + xg(t))}{1 - F(t)} = \frac{\frac{1}{\sqrt{2\pi}}.\frac{1}{t + xg(t)}.e^{-\frac{(t+xg(t))^2}{2}}}{\frac{1}{\sqrt{2\pi}}.\frac{1}{t}.e^{-t^2/2}}, \quad t \to \infty,$$

$$= \frac{t}{t+xg(t)}e^{-\frac{1}{2}(xg(t))^2 - xtg(t)}.$$

For $g(t) = \frac{1}{t}$, we get

$$\frac{1 - F(t + xg(t))}{1 - F(t)} = \frac{t}{t + x.\frac{1}{t}}e^{-\frac{1}{2}(x.\frac{1}{t})^2 - x} = \frac{1}{1 + x.\frac{1}{t^2}}e^{-x-\frac{1}{2}x^2.\frac{1}{t^2}} \to e^{-x}, \quad t \to \infty$$

Using Theorem **1.5.2** in chapter one, we conclude that the distribution function $F(x)$ belongs to the domain of attraction of the function $G_0(x)$, and we have the type (I) of extreme value distribution, i.e. there exist constants $a_n$and b_n, such that the following equality holds true:

$$\lim_{n\to\infty} P\left\{M_n \le \frac{x}{a_n} + b_n\right\} = \exp(-e^{-x}).$$

We now determine the constants $a_n$and b_n.

Let us first determine the constant u_n, such that $1 - F(u_n) \sim \frac{1}{n}e^{-x}$ as $n \to$, i.e.

$$1 - F(u_n) \sim \frac{\varphi(u_n)}{u_n},$$

$$\frac{\varphi(u_n)}{u_n} \sim \frac{1}{n}e^{-x}, \text{as } n \to \quad . \qquad (2.2)$$

Asymptotic relation (2.2) can be transformed in the following way:

$$\frac{1}{n}e^{-x}.\frac{u_n}{\varphi(u_n)} \to 1,$$

$\ln\left(\frac{1}{n}\ e^{-x}.\frac{u_n}{\varphi(u_n)}\right) \to \ln(1),$

$-\ln n - x + \ln u_n - \ln\varphi(u_n) \to 0,$ (2.3)

But $\ln\varphi(u_n) = \ln\left(\frac{1}{\sqrt{2\pi}}e^{-u_n^2/2}\right) = -\frac{1}{2}\ln 2\pi - \frac{u_n^2}{2}.$ (2.4)

Now substitute (2.4) into (2.3) to get

$-\ln n - x + \ln u_n + \frac{1}{2}\ln 2\pi + \frac{u_n^2}{2} \to 0.$ (2.5)

It follows from (2.5) that $\frac{u_n^2}{(2\ln n)} \to 1$ as $n \to$, and

$\ln u_n = \frac{1}{2}(\ln 2 + \ln\ln n) + o(1).$ (2.6)

The relation (2.5) can also be written in the form

$\frac{u_n^2}{2} = x + \ln n - \frac{1}{2}\ln 2\pi - \ln u_n + o(1).$ (2.7)

Now substitute the value of $\ln u_n$ from (2.6) into (2.7). We obtain

$\frac{u_n^2}{2} = x + \ln n - \frac{1}{2}\ln 2\pi - \frac{1}{2}\ln 2 - \frac{1}{2}\ln\ln n + o(1),$

$$\frac{u_n^2}{2} = x + \ln n - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n + o(1),$$

$u_n^2 = 2\ln n\left\{1 + \frac{x - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n}{\ln n} + o\left(\frac{1}{\ln n}\right)\right\}.$

Using the formula $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$ as $x \to$, we get

$$u_n = \sqrt{2\ln n}\left\{1 + \frac{1}{2\ln n}\left(x - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n\right) + o\left(\frac{1}{\ln n}\right)\right\},$$

$$u_n = \frac{x}{\sqrt{2\ln n}} + \sqrt{2\ln n} - \frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}} + o\left(\frac{1}{\sqrt{\ln n}}\right).$$

Since $u_n \sim \frac{x}{a_n} + b_n$, as $x \to$, we have

$$a_n = \sqrt{2\ln n}, \quad b_n = \sqrt{2\ln n} - \frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}}.$$

The following Theorem from Mladenović, P. [20].

Theorem 2.2.2. *Let* (Z_n) *be a sequence of independent random variables such that*

$$Z_n \epsilon \begin{cases} \mathcal{N}(\mu_1, \sigma_1^2), & with\ probability\ p, \\ \mathcal{N}(\mu_2, \sigma_2^2), & with\ probability\ q, \end{cases} \quad \textit{for all n,}$$

$Where\ p + q = 1.\ Let\ us\ denote\ M_n^* = max\{Z_1, \dots, Z_n\}.$ *if*

(a) $\sigma_1 > \sigma_2,\ \mu_1, \mu_2 \epsilon R\ \ or$
(b) $\sigma_1 = \sigma_2\ and\ \mu_1 > \mu_2,$

then for every real number x *the equality*

$$\lim_{n\to\infty} P\left\{M_n^* \le \frac{x}{a_n^*} + b_n^*\right\} = exp(-e^{-x}),$$

holds true, where the constants a_n^* *and* b_n^* *are given by*

$$a_n^* = \frac{\sqrt{2\,ln\,n}}{\sigma_1}, \quad b_n^* = \mu_1 + \sigma_1\sqrt{2\,ln\,n} - \frac{\sigma_1}{2\sqrt{2\,ln\,n}}\left(ln\,ln\,n + ln\frac{4\pi}{p^2}\right).$$

For the proof see Mladenović, P. [20].

2.3. Mixture of Cauchy distributions

Example (2.3.1):

Let (X_n) be a sequence of independent random variables with the Cauchy distribution $K(1,0)$, determined by the distribution function

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x.$$

Let $M_n = \max\{X_1, X_2, \dots, X_n\} = \max_{1\le i\le n} X_i$. For $x > 0$, we have

$$\frac{1 - F(tx)}{1 - F(t)} = \frac{1 - \frac{1}{2} - \frac{1}{\pi}\tan^{-1}(tx)}{1 - \frac{1}{2} - \frac{1}{\pi}\tan^{-1} t} = \frac{\frac{1}{2} - \frac{1}{\pi}\tan^{-1}(tx)}{\frac{1}{2} - \frac{1}{\pi}\tan^{-1} t} = \frac{\frac{\pi}{2} - \tan^{-1}(tx)}{\frac{\pi}{2} - \tan^{-1} t} \to \frac{1}{x}, \quad t \to \infty.$$

In fact, on substituting $\tan^{-1}(tx) = \frac{\pi}{2} - u$, we obtain $tx = \tan\left(\frac{\pi}{2} - u\right) = \cot u$,

Then $t = \frac{1}{x}\cot u$,

$$\lim_{t\to\infty}\left(\frac{\pi}{2} - \tan^{-1}(tx)\right).t = \lim_{u\to 0}\left(u.\frac{1}{x}\cot u\right) = \frac{1}{x}\lim_{u\to 0}(u.\cot u) = \frac{1}{x}.1 = \frac{1}{x}. \qquad (2.8)$$

And similarly, $\lim_{t\to\infty}\left(\frac{\pi}{2} - \tan^{-1}(t)\right).t = 1,$ (2.9)

From (2.8) and (2.9), we get

$$\frac{1-F(tx)}{1-F(t)} = \frac{\frac{1}{x}}{1} = \frac{1}{x}, t \to \infty.$$

Hence, the distribution function $F(x)$ belongs to the domain of attraction of the function $G_1(x)$, and we have the type (II) of extreme value distribution.

The normalizing constants are $a_n = \frac{1}{\gamma_n}$ *and* $b_n = 0$, where the constant γ_n can be computed from the equality $1 - F(\gamma_n) = 1 - \frac{1}{2} - \frac{1}{\pi}\tan^{-1}(\gamma_n) = \frac{1}{2} - \frac{1}{\pi}\tan^{-1}(\gamma_n) = \frac{1}{n}$.

Therefore, $\gamma_n = \tan(\frac{\pi}{2} - \frac{\pi}{n}) = \cot\frac{\pi}{n}$,

where $\tan(\frac{\pi}{2} - \frac{\pi}{n}) = \frac{\tan\frac{\pi}{2} - \tan\frac{\pi}{n}}{1+\tan\frac{\pi}{2}\tan\frac{\pi}{n}}$, *and* $\sin\frac{\pi}{2} = 1$, $\cos\frac{\pi}{2} = 0$.

For $x > 0$ we have $\lim_{n\to\infty} P\left\{M_n.\tan\frac{\pi}{n} \le x\right\} = e^{-x^{-1}}$.

For more information see Mladenović, P., [20].

Theorem 2.3.2. *Let $K(\lambda_i, 0)$ be the class of random variables with the distribution function $F_i(x) = \frac{1}{2} + \frac{1}{\pi}tan^{-1}\frac{x}{\lambda_i}$, $i = 1, 2$. Let (Z_n) be a sequence of independent random variables such that for every n,*

$$Z_n \epsilon \begin{cases} K(\lambda_1, 0), & \textit{with probability } p, \\ K(\lambda_2, 0), & \textit{with probability } q, \end{cases}$$

where $p + q = 1$, and $M_n^* = max\{Z_1, \dots, Z_n\}$. Then, for all $x > 0$ we have

$$\lim_{n\to\infty} P\left\{\frac{\pi}{n(p\lambda_1 + q\lambda_2)} M_n^* \le x\right\} = e^{-x^{-1}}.$$

Using Theorem **1.5.2** in chapter one, we conclude that the distribution function $F(x)$ belongs to the domain of attraction of the function $G_1(x)$, and we have the type (II) of extreme value distribution. The normalizing constants are $a_n^* = \frac{1}{\gamma_n}$ and $b_n^* = 0$, where $1 - F(\gamma_n) = \frac{1}{n}$, and $\gamma_n \sim \frac{n(p\lambda_1 + q\lambda_2)}{\pi}$. For the proof see Mladenović, P., [20].

2.4. Stable distributions

2.4.1. Introduction

Stable distributions are a rich class of probability distributions that allow skewness and heavy tails and have many interesting properties. In probability theory, a random variable is said to be stable distributed if it has the property that a linear combination of two independent copies of the variable has the same distribution. The stable distribution family is also sometimes referred to as the Levy alpha-stable distribution. The general stable distribution requires four parameters for complete description: $S_\alpha(\sigma, \beta, \mu)$, where $\alpha \epsilon (0,2]$ is an index of stability and also called the tail index, tail exponent or characteristic exponent, a skewness parameter $\beta \epsilon [-1,1]$, a scale parameter $\sigma > 0$ and a location parameter $\mu \epsilon R$. We shall use the following abbreviation $S_\alpha(\sigma, \beta, \mu)$ for this stable distribution.

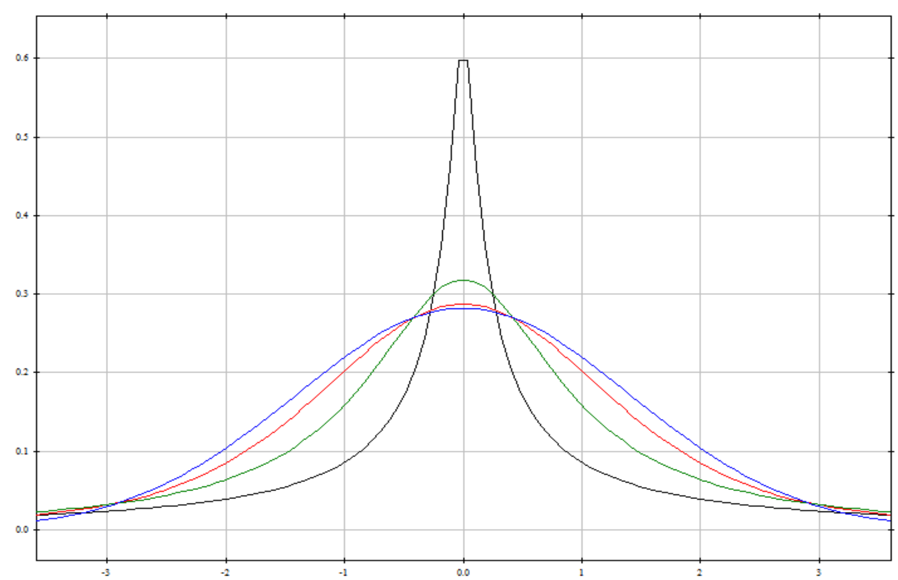

Figure (2.1): *Probability Density Function when* $(\alpha = 2{,}1.5{,}1{,}0.5, \beta = 0, \sigma = 1, \mu = 0)$

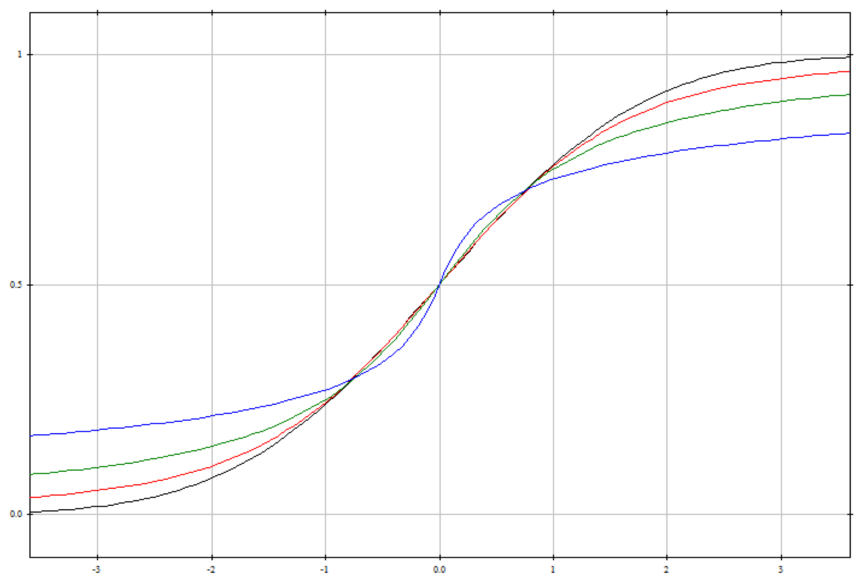

Figure (2.2): *Distribution Function when* $(\alpha = 2{,}1.5{,}1{,}0.5, \beta = 0, \sigma = 1, \mu = 0)$.

The following Theorems and Definitions are from [27].

Here we give four equivalent definitions of a stable distribution.
The first two definitions explain why these distributions are called *stable*, the third definition related it with the central limit theorem, and the fourth definition specifies the characteristic function of a stable random variable.

Definition 2.4.2. A random variable *X* is called *stable* if for any positive numbers *A* and *B*, there is a positive number *C* and a real number *D* such that

$$AX_1 + BX_2 =^d CX + D, \tag{2.10}$$

where X_1 and X_2 are independent copies of *X*, and where " $=^d$ " denotes equality in distribution.

Remark

(i) If equation (2.10) holds for *D=0*, then it is called *strictly stable.*

(ii) If $X =^d - X$, then it is called *symmetric stable*.

Theorem 2.4.3. *For any stable random variable X, there is a number* $\alpha \epsilon (0,2]$ *such that the number C in (2.1) satisfies*

$$C^{\alpha} = A^{\alpha} + B^{\alpha}, \tag{2.11}$$

where α *is called the index of stability or characteristic exponent.*

Definition 2.4.4 (equivalent to definition 2.4.2). A random variable *X* is called *stable* if for any $n \geq 2$, there is a positive number C_n and a real number d_n such that

$$X_1 + X_2 + \cdots + X_n \overset{"d"}{=} C_n x + d_n, \tag{2.12}$$

where X_i are independent copies of X, and $\overset{"d"}{=}$ denotes equality in distribution. X is *strictly stable* if and only if $d_n = 0$ for all n.

Remark

- **(i)** The first definition displays continuous combinations of two independent identically distributed random variables, while the second definition displays the sum of any number of independent identically distributed random variables.
- **(ii)** If equation (2.12) holds, then $C_n = n^{\frac{1}{\alpha}}$, for some $\alpha \epsilon (0,2]$.

Definition 2.4.5 (Equivalent to definitions 2.4.2and 2.4.4). A random variable *X* is called *stable* if it has a domain of attraction, i.e., if there exists a sequence of independent identically distributed random variables $Y_1, Y_2, \ldots,$ and sequences of positive numbers d_n and real numbers a_n such that

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{d_n} + a_n \to^{d} X, \tag{2.13}$$

where $\to^{d}$denotes convergence in distribution.

Remark

- **(i)** If *X* is Gaussian, and Y_i are independent identically distributed (i.i.d.) with finite variance, then equation (2.13) is just the central limit theorem.
- **(ii)** When $d_n = n^{\frac{1}{\alpha}}$, Y is said to belong to the "normal" domain of attraction *X*.

Generally, $d_n = n^{\frac{1}{\alpha}} L(n)$, where *L(x)*, $x > 0$, is a slowly varying function at infinity.

Definition 2.4.6 (equivalent to definitions 2.4.2, 2.4.4 and 2.4.5). A random variable X is called *stable* if there exists, $0 < \alpha \leq 2$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, μ is a real number such that the characteristic function of stable distribution has the following form:

$$Eexp(i\theta X) = \begin{cases} \exp\{-\sigma^{\alpha}|\theta|^{\alpha}(1 - i\beta(sign\theta)\tan\frac{\pi\alpha}{2}) + i\mu\theta\} & if\ \alpha \neq 1, \\ \exp\{-\sigma|\theta|(1 + i\beta\frac{2}{\pi}(sign\theta)\ln|\theta|) + i\mu\theta\} & if\ \alpha = 1, \end{cases} \tag{2.14}$$

and

$$sign\,\theta = \begin{cases} 1 & if\ \theta > 0, \\ 0 & if\ \theta = 0, \\ -1 & if\ \theta < 0. \end{cases}$$

Remark 2.4.7. Since (2.14) is characterized by four parameters, $\alpha\epsilon(0,2]$, $\sigma \geq 0, \beta\epsilon[-1,1], \mu\epsilon R$, we will denote stable distributions by $S_{\alpha}(\sigma, \beta, \mu)$ and write $X{\sim}S_{\alpha}(\sigma, \beta, \mu)$.

Remark 2.4.8. When $\alpha = 2$, the characteristic function (2.14) becomes $Eexp(i\theta X) = exp(i\mu\theta - \sigma^2\theta^2)$. This is the characteristic function of a Gaussian random variable with mean μ and variance $2\sigma^2$.

Remark 2.4.9. There are only three special cases in which a closed form expression is known for stable probability density function. These are the Gaussian case ($\alpha = 2, \beta = 0$), Cauchy case ($\alpha = 1, \beta = 0$), and Levy case ($\alpha = 0.5, \beta = \pm 1$) with the following densities:

(i) The Gaussian distribution $S_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2)$, whose density is

$$f(x) = \frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(x-\mu)^2}{4\sigma^2}}, \quad -\infty < x < \infty.$$

The distribution function, for which there is no closed form expression, is

$F(x) = P(X \leq x) = \Phi\left(\frac{(x-\mu)}{\sigma}\right)$, where $\Phi(Z) =$ Probability that a standard normal random variable is less than or equal Z.

(ii) The Cauchy distribution $S_1(\sigma, 0, \mu)$, whose density is

$$f(x) = \frac{\sigma}{\pi((x-\mu)^2 + \sigma^2)}, \quad -\infty < x < \infty.$$

(iii) The Levy distribution $S_{0.5}(\sigma, 1, \mu)$, whose density is

$$f(x) = \frac{\sqrt{\sigma}}{\sqrt{2\pi}(x-\mu)^{\frac{3}{2}}} e^{-\frac{\sigma}{2(x-\mu)}} , \quad \mu < x < \infty .$$

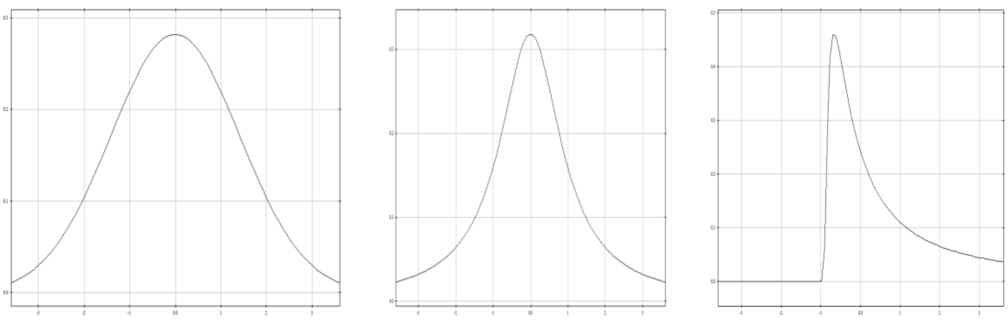

Figure (2.3): *Gaussian density functions when* $(\alpha = 2, \beta = 0)$, *Cauchy when* $(\alpha = 1, \beta = 0)$, *and Levy when* $(\alpha = 0.5, \beta = \pm 1)$, *respectively from left to right.*

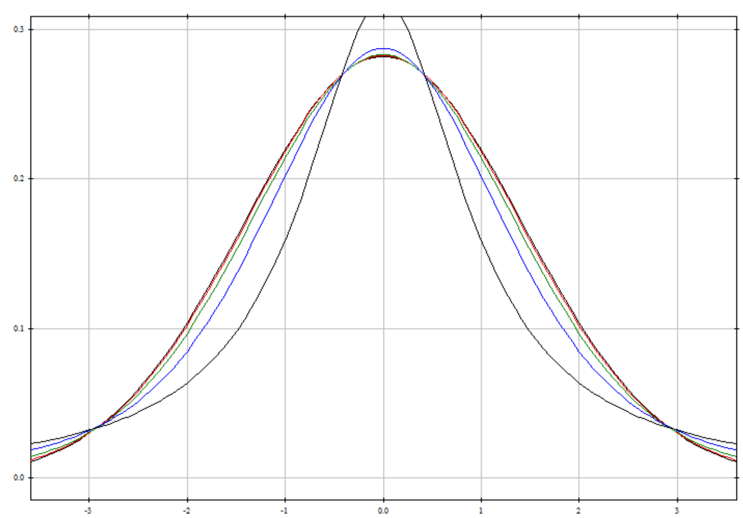

Figure (2.4): *Stable densities in the* $S_\alpha(1,0,0)$, *parameterization,*

$(\alpha = 1, 1.5, 1.8, 1.95, 2)$.

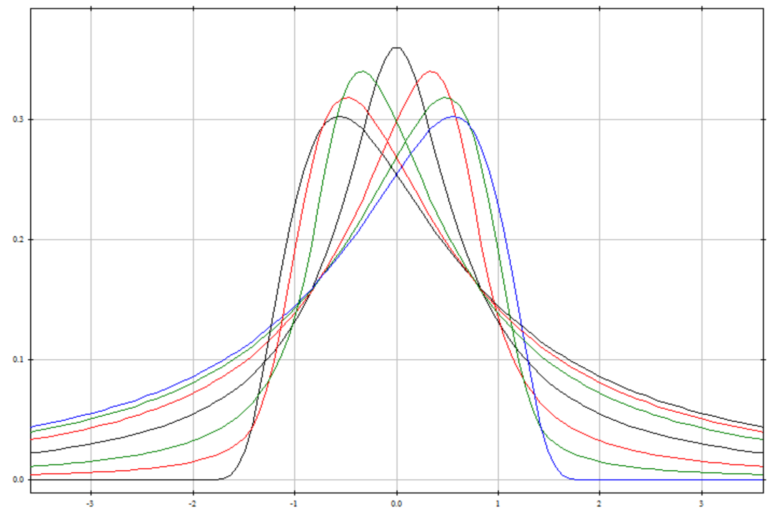

Figure (2.5): *Stable densities in the* $S_{0.8}(1, \beta, 0)$, *parameterization*,

$(\beta = -1, -0.8, -0.5, 0, 0.5, 0.8, \quad and\ 1)$.

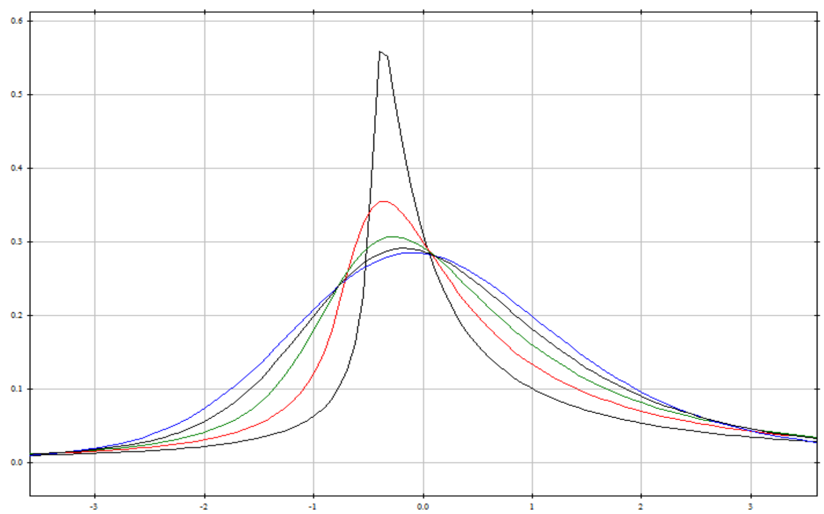

Figure (2.6): *Stable densities in the* $S_{\alpha}(1, 0.5, 0)$, *parameterization,*

$(\alpha = 0.5, 0.75, 1, 1.25, 1.5)$.

2.5. Properties of stable random variables:

The following properties are from [27].

Property 2.5.1. Let $X_1 and\ X_2$ be independent random variables with $X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i)$, $i = 1,2$. Then $X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu)$, with

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{\frac{1}{\alpha}}, \qquad \beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \qquad \mu = \mu_1 + \mu_2.$$

Proof: Use equation (2.14) and first we verify this for $\alpha \neq 1$. By independence,
$\ln Eexpi\theta(X_1 + X_2) = \ln(Eexpi\theta X_1) + \ln(Eexpi\theta X_2)$,

$$\ln(Eexpi\theta X_1) = -\sigma_1^\alpha|\theta|^\alpha(1 - i\beta_1(sign\theta)\tan\frac{\pi\alpha}{2}) + i\mu_1\theta \,, \qquad (2.15)$$

$$\ln(Eexpi\theta X_2) = -\sigma_2^\alpha|\theta|^\alpha(1 - i\beta_2(sign\theta)\tan\frac{\pi\alpha}{2}) + i\mu_2\theta \,, \qquad (2.16)$$

equation (2.15) + equation (2.16), then we get

$$= -(\sigma_1^\alpha + \sigma_2^\alpha)\,|\theta|^\alpha + i|\theta|^\alpha(\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha)\,sign\theta\tan\frac{\pi\alpha}{2} + i\theta(\mu_1 + \mu_2),$$

$$= -(\sigma_1^\alpha + \sigma_2^\alpha)\,|\theta|^\alpha\left\{1 - i\frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}sign\theta\tan\frac{\pi\alpha}{2}\right\} + i\theta(\mu_1 + \mu_2),$$

then, $\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{\frac{1}{\alpha}}$, $\beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}$, $\mu = \mu_1 + \mu_2$.

Second we verify for $\alpha = 1$. By independence,

$$\ln(Eexpi\theta X_1) = -\sigma_1|\theta|(1 + i\beta_1\frac{2}{\pi}(sign\theta)\ln|\theta|) + i\mu_1\theta, \qquad (2.17)$$

$$\ln(Eexpi\theta X_2) = -\sigma_2|\theta|(1 + i\beta_2\frac{2}{\pi}(sign\theta)\ln|\theta|) + i\mu_2\theta, \qquad (2.18)$$

then equation (2.17) + equation (2.18), we get

$$= -(\sigma_1 + \sigma_2)|\theta|\left\{1 + i\frac{\beta_1\sigma_1 + \beta_2\sigma_2}{\sigma_1 + \sigma_2}\frac{2}{\pi}(sign\theta)\ln|\theta|\right\} + i\theta(\mu_1 + \mu_2),$$

then, $\sigma = \sigma_1 + \sigma_2$, $\beta = \frac{\beta_1\sigma_1 + \beta_2\sigma_2}{\sigma_1 + \sigma_2}$, $\mu = \mu_1 + \mu_2$.

Property 2.5.2. Let $X \sim S_\alpha(\sigma, \beta, \mu)$ and let (a) be a real constant. Then

$$X + a \sim S_\alpha(\sigma, \beta, \mu + a).$$

Proof:

(i) If $\alpha \neq 1$, then

$$\ln Eexpi\theta(X+a) = \ln(Eexpi\theta X) + \ln(Eexpi\theta a),$$

But $\ln(Eexpi\theta X) = -\sigma^{\alpha}|\theta|^{\alpha}(1 - i\beta(sign\theta)\tan\frac{\pi\alpha}{2}) + i\mu\theta$, (2.19)

and $\ln(Eexpi\theta a) = ia\theta$, (2.20)

because $Eexpi\theta a = E\left(e^{i\theta a}\right) = \sum_n P_n e^{i\theta a} = e^{i\theta a}\sum_n P_n = e^{i\theta a}.1 = e^{i\theta a}$.

Then equation (2.19) + equation (2.20), we get

$$\ln Eexpi\theta(X+a) = -\,\sigma^{\alpha}|\theta|^{\alpha}(1 - i\beta(sign\theta)\tan\frac{\pi\alpha}{2}) + i(\mu + a)\theta, \qquad if\ \ \alpha \neq 1.$$

(ii) If $\alpha = 1$, then

$$\ln Eexpi\theta(X+a) = -\sigma|\theta|(1 + i\beta\tfrac{2}{\pi}(sign\theta)\ln|\theta|) + i\mu\theta + ia\theta,$$

$$\ln Eexpi\theta(X+a) = -\sigma|\theta|(1 + i\beta\frac{2}{\pi}(sign\theta)\ln|\theta|) + i\theta(\mu + a), \qquad if\ \alpha = 1.$$

Then, $X + a \sim S_{\alpha}(\sigma, \beta, \mu + a)$.

Property 2.5.3. Let $X \sim S_{\alpha}(\sigma, \beta, \mu)$ and let (a) be a non-zero real constant. Then

$$aX \sim S_{\alpha}(|a|\sigma, sign(a)\beta, a\mu), \qquad if\ \alpha \neq 1,$$

$$aX \sim S_1\left(|a|\sigma, sign(a)\beta, a\mu - \frac{2}{\pi}a(\ln|a|)\sigma\beta\right), \qquad if\ \alpha = 1.$$

Proof:

(i) if $\alpha \neq 1$, then

$$\ln Eexpi\theta(aX) = -\,\sigma^{\alpha}|\theta a|^{\alpha}(1 - i\beta(sign(a\theta))\tan\frac{\pi\alpha}{2}) + i\mu(a\theta),$$

$$\ln Eexpi\theta(aX) = -\,(\sigma|a|)^{\alpha}|\theta|^{\alpha}(1 - i\beta(sign(a)sign(\theta))\tan\frac{\pi\alpha}{2}) + i(\mu a)\theta,$$

then

$$aX \sim S_{\alpha}(|a|\sigma, sign(a)\beta, a\mu), \qquad if\ \alpha \neq 1.$$

(ii) If $\alpha = 1$, then

$$\ln Eexpi\theta(aX) = -\sigma|\theta a|(1 + i\beta\frac{2}{\pi}(sign(a\theta))\ln|a\theta|) + i\mu(a\theta),$$

$$\ln Eexpi\theta(aX) = -|a|\sigma|\theta|(1 + i\beta\frac{2}{\pi}sign(a)sign(\theta)\{\ln|a| + \ln|\theta|\}) + i\mu(a\theta),$$

$$= -|a|\sigma|\theta|(1 + i\beta\frac{2}{\pi}sign(a)sign(\theta)\ln|\theta|) + i\left(\mu a - \beta\frac{2}{\pi}|a||\theta|\sigma.sign(a).\ln|a|\,sign(\theta)\right)\theta\ ,$$

then

$$aX \sim S_1\left(|a|\sigma, sign(a)\beta, a\mu - \frac{2}{\pi}|a|(\ln|a|)\sigma\beta.sign(a)\right), \qquad if\ \alpha = 1.$$

Property 2.5.4. For any $0 < \alpha < 2$,

$X \sim S_\alpha(\sigma,\beta,0) \Leftrightarrow -X \sim S_\alpha(\sigma,-\beta,0)$.

Proof :

(i) $\ln Eexpi\theta X = -\sigma^\alpha|\theta|^\alpha(1 - i\beta(sign(\theta))\tan\frac{\pi\alpha}{2}) + i\mu\theta$,

but $S_\alpha(\sigma,\beta,0) = -\sigma^\alpha|\theta|^\alpha(1 - i\beta(sign(\theta))\tan\frac{\pi\alpha}{2})$, if $\alpha \neq 1$,

and $S_\alpha(\sigma,\beta,0) = -\sigma|\theta|(1 + i\beta\frac{2}{\pi}sign\theta\ln|\theta|)$, if $\alpha = 1$,

then $X \sim S_\alpha(\sigma,\beta,0)$.

(ii) $S_\alpha(\sigma,-\beta,0) = -\sigma^\alpha|\theta|^\alpha(1 + i\beta(sign\theta))\tan\frac{\pi\alpha}{2})$, if $\alpha \neq 1$,

$$= -\left\{\sigma^\alpha|\theta|^\alpha(1 + i\beta(sign\theta))\tan\frac{\pi\alpha}{2})\right\},$$

and $S_\alpha(\sigma,-\beta,0) = -\sigma|\theta|(1 - i\beta\frac{2}{\pi}sign\theta\ln|\theta|)$, if $\alpha = 1$,

$$= -\left\{\sigma|\theta|(1 - i\beta\frac{2}{\pi}sign\theta\ln|\theta|)\right\},$$

then $-X \sim S_\alpha(\sigma,-\beta,0)$,

from (i) and (ii) then we get $X \sim S_\alpha(\sigma,\beta,0) \Leftrightarrow -X \sim S_\alpha(\sigma,-\beta,0)$.

Remark in property 2.5.4. The distribution $S_\alpha(\sigma,\beta,0)$ is said to be skewed to the right if $\beta > 0$ and to the left if $\beta < 0$. It is said to be totally skewed to the right if $\beta = 1$ and totally skewed to the left if $\beta = -1$.

Property 2.5.5. $X \sim S_\alpha(\sigma,\beta,\mu)$ is *symmetric* if and only if $\beta = 0$ *and* $\mu = 0$. It is symmetric about μ if and only if $\beta = 0$.

Proof: For a random variable to be symmetric, it is necessary and sufficient that its characteristic function be real. By (2.14), when $\beta = 0, \mu = 0$ then

$\ln E\exp i\theta X = -\sigma^\alpha|\theta|^\alpha,$ if $\alpha \neq 1,$

$\ln E\exp i\theta X = -\sigma|\theta|,$ if $\alpha = 1.$

Remark 2.5.6. Asymmetric stable random variable is strictly stable, but a strictly stable random variable is not necessarily symmetric.

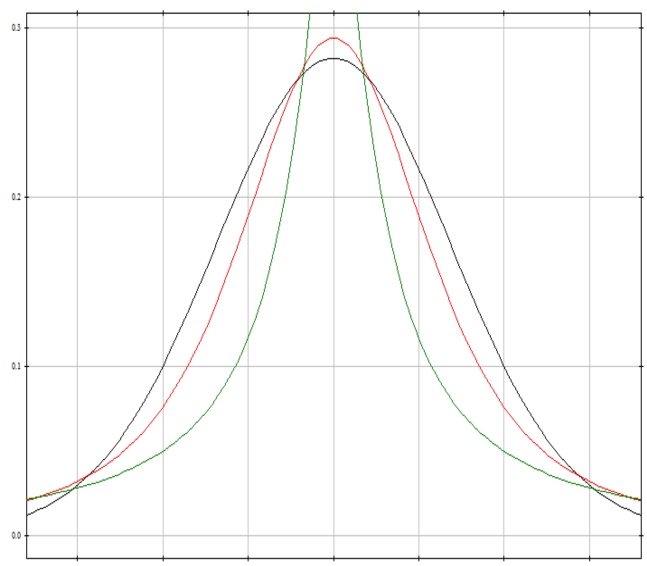

Figure (2.7): *Symmetric stable densities* for $Z \sim S_\alpha(1,0,0)$, $\alpha = (0.7, 1.3, 1.9)$.

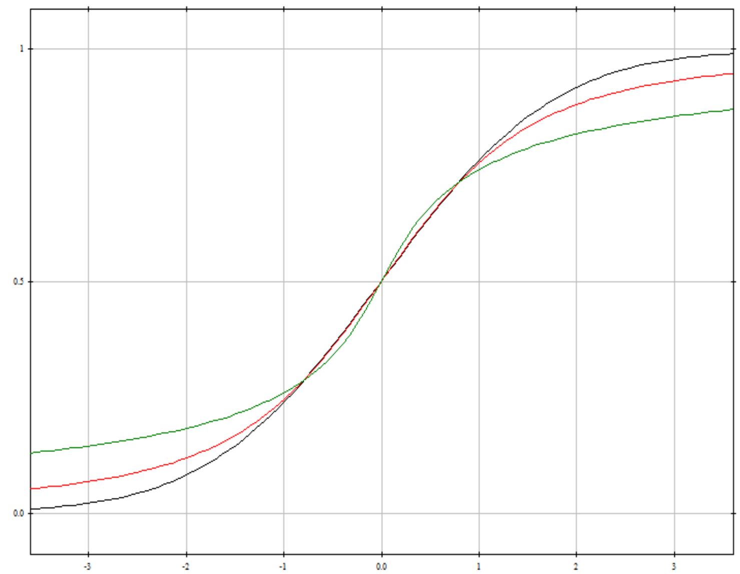

Figure (2.8): *Symmetric stable distribution functions* for $Z \sim S_\alpha(1,0,0)$, $\alpha = (0.7, 1.3, 1.9)$.

Property 2.5.7. Let $X \sim S_\alpha(\sigma, \beta, \mu)$, with $\alpha \neq 1$. Then X is *strictly stable* if and only if $\mu = 0$.
Proof: let X_1, X_2 be independent copies of X and let A and B be arbitrary positive constants.
By properties (2.5.1) and (2.5.3),
$AX_1 + BX_2 \sim S_\alpha\left(\sigma(A^\alpha + B^\alpha)^{\frac{1}{\alpha}}, \beta, \mu(A+B)\right)$. We must set $C = (A^\alpha + B^\alpha)^{\frac{1}{\alpha}}$ in (2.10) by properties (2.5.2) and (2.5.3),

$$CX + D \sim S_\alpha\left(\sigma(A^\alpha + B^\alpha)^{\frac{1}{\alpha}}, \beta, \mu(A^\alpha + B^\alpha)^{\frac{1}{\alpha}} + D\right),$$

and therefore, we have $AX_1 + BX_2 =^d CX + D \; with \; D = 0 \; iff \; \mu = 0$.

Corollary 2.5.8. Let $X \sim S_\alpha(\sigma, \beta, \mu)$, with $\alpha \neq 1$. Then $X - \mu$ is *strictly stable.*
Proof: use properties 2.5.2 and 2.5.7.

Remark 2.5.9. Thus, any alpha stable random variable with $\alpha \neq 1$ can be made strictly stable by shifting. This is not true when $\alpha = 1$.

Property 2.5.10. Let $X \sim S_\alpha(\sigma, \beta, \mu)$, with $\alpha = 1$. Then X is *strictly stable* if and only if $\beta = 0$.
Proof: let X_1, X_2 be independent copies of X and let $A > 0, B > 0$.

And use properties 2.5.3 and 2.5.1.

Corollary 2.5.11. If $X_1, X_2, \dots, X_n$ are independent identically distributed $S_\alpha(\sigma, \beta, \mu)$, then

$$X_1 + X_2 + \cdots + X_n =^d n^{\frac{1}{\alpha}} X_1 + \mu\left(n - n^{\frac{1}{\alpha}}\right), \qquad if\ \alpha \neq 1,$$

and

$$X_1 + X_2 + \cdots + X_n =^d nX_1 + \frac{2}{\pi}\sigma\beta, \qquad if\ \alpha = 1.$$

Remark 2.5.12. The random variable $X \sim S_\alpha(\sigma, 1, 0)$ with $0 < \alpha < 1$ is called a *stable subordinator.*

Proposition 2.5.13. The "Laplace transform" $Ee^{-\gamma X}, \gamma \geq 0$, of the random variable $X \sim S_\alpha(\sigma, 1, 0), 0 < \alpha \leq 2, \sigma \geq 0$, equals

$$Ee^{-\gamma X} = exp\left\{-\frac{\sigma^\alpha}{\cos\frac{\pi\alpha}{2}} \cdot \gamma^\alpha\right\} \qquad if\ \alpha \neq 1,$$

and

$$Ee^{-\gamma X} = exp\left\{\sigma . \frac{2}{\pi}\gamma \ln \gamma\right\} \qquad if\ \alpha = 1.$$

Remark 2.5.14. The constant $-\sigma^\alpha\left(\cos\frac{\pi\alpha}{2}\right)^{-1}$ is negative if $0 < \alpha < 1$, and is positive if $1 < \alpha \leq 2$. It equals σ^2 when $\alpha = 2$.

Property 2.5.15. Let *X* have distribution $S_\alpha(\sigma, \beta, 0)$ with $\alpha < 2$. Then there exist two independent identically distributed (i.i.d.) random variables $Y_1\ and\ Y_2$ with common distribution $S_\alpha(\sigma, 1, 0)$ such that

$$X =^d \left(\frac{1+\beta}{2}\right)^{\frac{1}{\alpha}} Y_1 - \left(\frac{1-\beta}{2}\right)^{\frac{1}{\alpha}} Y_2, \qquad if\ \alpha \neq 1,$$

and

$$X =^d \left(\frac{1+\beta}{2}\right) Y_1 - \left(\frac{1-\beta}{2}\right) Y_2 + \sigma\left(\frac{1+\beta}{\pi}\ln\frac{1+\beta}{2} - \frac{1-\beta}{\pi}\ln\frac{1-\beta}{2}\right), \qquad if\ \alpha = 1.$$

Proof use properties 2.5.1, 2.5.2 and 2.5.3, in [27].

2.6. Overview in infinitely divisible distributions

Stable distributions have a long history in the subject of probability. They form a subset of the class of so-called "*infinitely-divisible*" distributions, a class of characteristic functions at the heart of general central limit theory.

The following definitions and Theorems are from [13].

Definition 2.6.1. A distribution function $F(x)$ and the corresponding characteristic function $f(t)$ are said to be *infinitely divisible* if for every positive integer n there exist a characteristic function $f_n(t)$ such that $f(t) = (f_n(t))^n$ *then* $f_n(t) = \sqrt[n]{f(t)}$, (2.21)

Property 2.6.2. Stable distributions are infinitely divisible.

Examples 2.6.3. Infinitely divisible distributions include:

(i) The normal distribution with parameters (μ, σ^2) is infinitely divisible, because the characteristic function of the normal distribution has the form $f(t) = e^{i\mu t - \frac{\sigma^2 t^2}{2}}$, so that then for every positive integer *n* there exist a characteristic function $f_n(t)$ such that $f_n(t) = e^{i\frac{\mu}{n}t - \frac{1}{2}\left(\frac{\sigma}{\sqrt{n}}\right)^2 t^2}$ is the characteristic function of the normal distribution with parameters $\left(\frac{\mu}{n}, \frac{\sigma}{\sqrt{n}}\right)$.

(ii) The Poisson distribution with parameters (λ) is infinitely divisible, because the characteristic function of the Poisson distribution has the form $f(t) = e^{\lambda(e^{it}-1)}$, so that then for every positive integer *n* there exist a characteristic function $f_n(t)$ such that $f_n(t) = e^{\frac{\lambda}{n}(e^{it}-1)}$ is the characteristic function of the Poisson distribution with parameter $\left(\frac{\lambda}{n}\right)$.

(iii) Cauchy distribution and the "chi-squared" distribution.

Theorem 2.6.4. *The characteristic function of an infinitely divisible distribution never vanishes.*

Here we give an example of a discrete random variables taking the values -1, 0, 1, with probability $\frac{1}{8}, \frac{3}{4}, \frac{1}{8}$. Its characteristic function is

$$f(t) = E\left(e^{itx}\right) = \sum_n P_n\, e^{itx_n} = \frac{3 + \cos t}{4},$$

where $e^{it} = \cos t + i \sin t$ *and* $e^{-it} = \cos t - i \sin t$,

the function *f* is positive and therefore does not vanish.

Before the construction of the general theory two basic elementary types of such random functions were known:

(i) The normal type then the characteristic function $f_n(t)$ is given by the formula $\log f_n(t) = n\left(i\mu t - \frac{\sigma^2 t^2}{2}\right)$, (2.22)

(ii) The Poisson type then the characteristic function $f_n(t)$ is given by the formula $\log f_n(t) = n\lambda\left(e^{it} - 1\right)$, (2.23)

By combining (2.22) and (2.23) then we get the formula is

$$\log f_n(t) = \mathrm{n}\left\{i\mu t - \frac{\sigma^2 t^2}{2} + \lambda \int_{-\infty}^{+\infty}\left(e^{itx} - 1\right) dF(x)\right\}, \quad (2.24)$$

$$\log f_n(t) = \mathrm{n}\left\{i\mu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itx} - 1\right) dM(x) + \int_{0}^{+\infty}\left(e^{itx} - 1\right) dN(x)\right\}, \quad (2.25)$$

where $\int_{-\infty}^{0}\left(e^{itx} - 1\right) dM(x) = \lim_{a\to 0} \int_{-\infty}^{a}\left(e^{itx} - 1\right) dM(x), \ \ a < 0,$ and

$$\int_{0}^{\infty}\left(e^{itx} - 1\right) dN(x) = \lim_{a\to 0} \int_{a}^{\infty}\left(e^{itx} - 1\right) dN(x), \ \ a > 0,$$

then $$\log f_n(t) = \mathrm{n}\left\{\begin{matrix} i\mu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itx} - 1 - itx\right) dM(x) \\ + \int_{0}^{+\infty}\left(e^{itx} - 1 - itx\right) dN(x) \end{matrix}\right\}, \quad (2.26)$$

$$\log f_n(t) = \mathrm{n}\left\{i\mu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM(x) + \int_{0}^{+\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dN(x)\right\}$$

and

$$\log f(t) = \left\{i\mu t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dM(x) + \int_{0}^{+\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dN(x)\right\}.$$

Theorem 2.6.5. The Levy-Khinchine canonical representation:

The function $f(t)$ is the characteristic function of an infinitely divisible distribution if and only if it can be written in the form:

$log\ f(t) = i\mu t + \int_{-\infty}^{\infty}\left[e^{itx} - 1 - \frac{itx}{1+x^2}\right]\frac{1+x^2}{x^2}dG(x)$, *where* μ *is a real constant,* $G(x)$ *is a non-decreasing and bounded function, such that* $G(-\infty) = 0$ *and the integral at* $x = 0$ *is equal* $\frac{-t^2}{2}$, *i.e.,* $\left\{\left[e^{itx} - 1 - \frac{itx}{1+x^2}\right]\frac{1+x^2}{x^2}\right\}_{x=0} = \frac{-t^2}{2}$.

Theorem 2.6.6. The Levy canonical representation:

The function $f(t)$ *is the characteristic function of an infinitely divisible distribution if and only if it can be written in the form:*

$$log\ f(t) = i\mu t - \frac{\sigma^2}{2}t^2 + \int_{-\infty}^{0}\left[e^{itx} - 1 - \frac{itx}{1+x^2}\right]dM(x) + \int_{0}^{\infty}\left[e^{itx} - 1 - \frac{itx}{1+x^2}\right]dN(x),$$

where μ *is a real constant,* σ^2 *is a real and non negative constant and the functions* $M(x), N(x)$ *satisfy the following conditions:*

(i) $M(x)$ *and* $N(x)$ *are non-decreasing in* $(-\infty, 0)$ *and* $(0, +\infty)$.

(ii) $M(-\infty) = N(+\infty) = 0$.

(iii) *The integrals* $\int_{-\varepsilon}^{0} x^2\, dM(x) + \int_{0}^{\varepsilon} x^2\, dN(x)$ *are finite for every* $\varepsilon > 0$.

Theorem 2.6.7. The Kolmogorov canonical representation:

The function $f(t)$ *is the characteristic function of an infinitely divisible distribution with finite second moment iff it can be written in the form:*

$log\ f(t) = i\mu t + \int_{-\infty}^{\infty}\left[e^{itx} - 1 - itx\right]\frac{dK(x)}{x^2}$, *where* μ *is a real constant,* $K(x)$ *is a non-decreasing and bounded function, such that* $K(-\infty) = 0$ *and the integral at* $x = 0$ *is equal* $\frac{-t^2}{2}$, *i.e.,* $\left\{\left[e^{itx} - 1 - itx\right]\frac{1}{x^2}\right\}_{x=0} = \frac{-t^2}{2}$.

2.7. Mixtures of stable distributions

In this chapter we consider the mixtures of stable distributions [6]. This paper extends the result of Mladenović, P., [20].

Our result will make use of the two Theorems from [17] (for general reference, see also [25, 24]).

The first Theorem (1.5.2) in chapter 1, which is Theorem 1.6.2 from [17], enables us to determine the domain of attraction and its type for our common distribution function.

The second Theorem (1.5.12) in chapter 1, which is Theorem 1.5.1 from [17], enables us to find the normalizing constants a_n and b_n in (1.2).

The following property 1.2.15 in [27, p.16 -18], will be useful in our proofs.

Property 2.7.1. Let $X \sim S_\alpha(\sigma, \beta, \mu)\ with\ 0 < \alpha < 2$. Then

$$\begin{cases} \lim\limits_{x\to} x^\alpha P\{X > x\} = C_\alpha \dfrac{1+\beta}{2}\sigma^\alpha, \\ \lim\limits_{x\to} x^\alpha P\{X < -x\} = C_\alpha \dfrac{1-\beta}{2}\sigma^\alpha, \end{cases}$$

where $C_\alpha = \left(\int_0 x^{-\alpha} \sin x dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{(2-\alpha)\cos(\pi\alpha/2)} & if\ \alpha \neq 1, \\ 2/\pi & if\ \alpha = 1. \end{cases}$

As a special case, if $X \sim S_\alpha(\sigma, 0,0), then\ as\ x \to\ ,$

$$P(X > x) \sim \sigma^\alpha \tfrac{C_\alpha}{2} x^{-\alpha}.$$

Suppose now $X \sim S_\alpha(\sigma, -1,0)$. Since $\beta = -1$, property 2.7.1 gives $\lim_{x\to}\ x^\alpha P(X > x) = 0, i.e.,$ $P(X > x)$ tends to 0 faster than $x^{-\alpha}$ as $x \to$.

When $\alpha > 1, \beta = -1\ as\ x \to\ ,$

$$P(X > x) \sim \frac{1}{\sqrt{2\pi\alpha(\alpha-1)}}\left(\frac{x}{\alpha\widehat{\sigma_\alpha}}\right)^{\frac{-\alpha}{2(\alpha-1)}} exp\left(-(\alpha-1)\left(\frac{x}{\alpha\widehat{\sigma_\alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right), \tag{2.27}$$

where $\widehat{\sigma_\alpha} = \sigma\left(\cos\frac{\pi}{2}(2-\alpha)\right)^{\frac{-1}{\alpha}}$.

When $\alpha = 1$,

$$P(X > x) \sim \frac{1}{\sqrt{2\pi}} exp\left(-\frac{(\pi/2\sigma)x - 1}{2} - e^{(\pi/2\sigma)x-1}\right).$$

Remark 2.7.2. When $1 - F_i(x) \sim x^{-\alpha_i} C_{\alpha_i} \frac{1+\beta}{2} \sigma^{\alpha_i}$, $i = 1,2$, then when $1 < \alpha_1 < \alpha_2$, the distribution $F_1(x)$ is one with the heavier right tail because $\frac{1-F_1(x)}{1-F_2(x)} \to \infty \; as \; x \to$.

However, in the case when $\beta = -1$ and the right tail of the distribution of F_1, F_2 is given by $P(X > x) \sim \frac{1}{\sqrt{2\pi\alpha(\alpha-1)}} \left(\frac{x}{\alpha\widehat{\sigma_\alpha}}\right)^{\frac{-\alpha}{2(\alpha-1)}} exp\left(-(\alpha - 1)\left(\frac{x}{\alpha\widehat{\sigma_\alpha}}\right)^{\frac{\alpha}{\alpha-1}}\right)$, with $1 < \alpha_1 < \alpha_2$

(Equation (1.2.11) from Samorodnotsky and Taqqu [27]), the situation is the opposite one, namely F_2 is the distribution with the heavier right tail, since we have $\frac{1-F_1(x)}{1-F_2(x)} \to 0 \; as \; x \to$. Theorem 2.7.3 reflects this property.

Main Results

We are now ready to determine the type of the domain of attraction and corresponding normalizing constants for our mixture.

THEOREM 2.7.2. *Let (X_n) be a sequence of independent random variables such that*

$$X_n \sim \begin{cases} S_\alpha(\sigma_1, 0,0), & with\, probability\; p, \\ S_\alpha(\sigma_2, 0,0), & with\, probability\; q, \end{cases} \quad for\; all\; n$$

where $p, q > 0$ and $X \sim S_\alpha(\sigma, 0,0)$ denotes the stable distribution with $P(X > x) \sim \sigma^\alpha \frac{C_\alpha}{2} x^{-\alpha}$ *and* $0 < \alpha < 2$, $\sigma_1 \neq \sigma_2$.

Let $M_n = max_{1 \le j \le n} X_j$. Then, the limiting distribution of M_n is given by

$P\{a_n(M_n - b_n) \le x\} \to exp(-x^{-\alpha})$, $n \to$,

where the normalizing constants a_n and b_n are given by

$a_n = (nC)^{\frac{-1}{\alpha}}$ *and* $b_n = 0$,

with $C = (pA + qB)$, $A = \frac{C_\alpha \sigma_1^\alpha}{2}$ *and* $B = \frac{C_\alpha \sigma_2^\alpha}{2}$.

The proof can be found in [8]. We give another proof in this chapter. The proof has two main parts. In the first part we obtain that the distribution function of M_n belongs to the domain of attraction of Fréchet distribution. After that we determine the normalizing constants.

Proof: The distribution function of the random variable X is given by $F(x) = pF_1(x) + qF_2(x)$ where $X_1 \sim S_\alpha(\sigma_1, 0,0)$ and $X_2 \sim S_\alpha(\sigma_2, 0,0)$.

Then $1 - F_1(x) = P\{X_1 > x\} \sim Ax^{-\alpha}$ and $1 - F_2(x) = P\{X_2 > x\} \sim Bx^{-\alpha}$,

where $A = \frac{C_\alpha \sigma_1^\alpha}{2}$ and $B = \frac{C_\alpha \sigma_2^\alpha}{2}$.

For the function $F(x) = pF_1(x) + qF_2(x)$, we obtain

$1 - F(x) \sim Cx^{-\alpha}$, where $C = (pA + qB)$, $x \to$.

We now consider the asymptotic behavior of the tail $1 - F(x)$, as $x \to$. $For\ x > 0$, we have

$$\lim_{t\to} \frac{1 - F(tx)}{1 - F(t)} = \lim_{t\to} \frac{C(tx)^{-\alpha}}{C(t)^{-\alpha}} = \lim_{t\to} x^{-\alpha} = x^{-\alpha}.$$

That's why, the distribution function $F(x)$ belongs to the domain of attraction of the function $G_1(x)$, and we have the type (II) of extreme value distribution, i.e. there exist constants a_n and b_n, such that the following equality holds true:

$$P\left\{M_n \le \frac{x}{a_n} + b_n\right\} \to \exp(-x^{-\alpha}).$$

We now determine the constants a_n and b_n.

Let us first determine the constant u_n, such that $1 - F(u_n) \sim \frac{1}{n}x^{-\alpha}$ as $n \to$, $i.e.$

$1 - pF_1(u_n) - qF_2(u_n) \sim \frac{1}{n}x^{-\alpha}$ as $n \to$. That means

$C(u_n)^{-\alpha} \sim \frac{1}{n}x^{-\alpha}$, $n \to$, and we obtain

$u_n \sim (nC)^{1/\alpha} x$, as $n \to$.

Using Theorem 1.5.12 we obtain

$$P\left\{M_n \le (nC)^{1/\alpha} x\right\} \to \exp(-x^{-\alpha}), \quad \text{as } n \to \quad . \tag{2.28}$$

$$\text{But } P\left\{M_n \le \frac{x}{a_n} + b_n\right\} \to G(x). \tag{2.29}$$

Now we compare the equation (2.28) with the equation (2.29). We obtain

$a_n = (nC)^{-1/\alpha}$ and $b_n = 0$,

where $C = (pA + qB), A = \frac{C_\alpha {\sigma_1}^\alpha}{2}$, $B = \frac{C_\alpha {\sigma_2}^\alpha}{2}$ $and\ G(x) = \exp(-x^{-\alpha})$. □

THEOREM 2.7.3. *Let (X_n) be a sequence of independent random variables such that*

$$X_n \sim \begin{cases} S_{\alpha_1}(\sigma,-1,0), & with\ probability\ p, \\ S_{\alpha_2}(\sigma,-1,0), & with\ probability\ q, \end{cases} \quad for\ all\ n \tag{2.30}$$

where $p, q > 0,\ 1 < \alpha_1 < \alpha_2$ and $X \sim S_\alpha(\sigma,-1,0)$ denotes the stable distribution with the tail behavior given by formula (2.27).

Let $M_n^* = max_{1\le j\le n} X_j$. Then the limiting distribution of M_n^* is given by

$$P\left\{M_n^* \le \frac{x}{a_n^*} + b_n^*\right\} \to exp(-e^{-x}), as\ n \to \infty,$$

where the normalizing constants are

$$a_n^* = \frac{\alpha_2}{\alpha_2-1}\,\frac{\ln n^{1/\alpha_2}}{B_2^{\frac{1-\alpha_2}{\alpha_2}}} \quad and \quad b_n^* = B_2^{\frac{1-\alpha_2}{\alpha_2}}\left\{\frac{1}{\ln n^{\frac{1-\alpha_2}{\alpha_2}}} + \frac{\alpha_2-1}{\alpha_2}\left(\frac{2\ln qA_2\sqrt{B_2} - \ln\ln n}{2\ln n^{1/\alpha_2}}\right)\right\},$$

where $A_2 = \frac{1}{\sqrt{2\pi\alpha_2(\alpha_2-1)}}\left(\alpha_2\widehat{\sigma_{\alpha_2}}\right)^{\frac{\alpha_2}{2(\alpha_2-1)}}$, $B_2 = (\alpha_2-1)\left(\alpha_2\widehat{\sigma_{\alpha_2}}\right)^{\frac{-\alpha_2}{\alpha_2-1}}$ *and*

$\widehat{\sigma_{\alpha_2}} = \sigma\left(\cos\frac{\pi}{2}(2-\alpha_2)\right)^{\frac{-1}{\alpha_2}}$.

In this case the proof has also two main parts. In the first part we obtain that the distribution function of M_n^* belongs to the domain of attraction of Gumbel distribution. After that we determine the normalizing constants.

Proof: We have that:

$$1 - F_j(x) = A_j x^{\frac{-\alpha_j}{2(\alpha_j-1)}} exp\left\{-B_j x^{\frac{\alpha_j}{\alpha_j-1}}\right\},$$

where $A_j = \frac{1}{\sqrt{2\pi\alpha_j(\alpha_j-1)}}\left(\alpha_j\widehat{\sigma_{\alpha_j}}\right)^{\frac{\alpha_j}{2(\alpha_j-1)}} > 0$ and $B_j = (\alpha_j-1)\left(\alpha_j\widehat{\sigma_{\alpha_j}}\right)^{\frac{-\alpha_j}{\alpha_j-1}} > 0$, j = 1, 2

$\widehat{\sigma_{\alpha_1}} = \sigma\left(\cos\frac{\pi}{2}(2-\alpha_1)\right)^{-1/\alpha_1}$ and $\widehat{\sigma_{\alpha_2}} = \sigma\left(\cos\frac{\pi}{2}(2-\alpha_2)\right)^{-1/\alpha_2}$.

For the distribution function $F(x) = pF_1(x) + qF_2(x)$, we obtain

$1-F(x)=p\big(1-F_1(x)\big)+q\big(1-F_2(x)\big),$

$1-F(x)=pA_1x^{\frac{-\alpha_1}{2(\alpha_1-1)}}\,exp\left\{-B_1x^{\frac{\alpha_1}{\alpha_1-1}}\right\}+qA_2x^{\frac{-\alpha_2}{2(\alpha_2-1)}}\,exp\left\{-B_2x^{\frac{\alpha_2}{\alpha_2-1}}\right\}.$

Then $1-F(x)=\ qA_2x^{\frac{-\alpha_2}{2(\alpha_2-1)}}\,exp\left\{-B_2x^{\frac{\alpha_2}{\alpha_2-1}}\right\}\left(\frac{pA_1}{qA_2}x^{\frac{\alpha_2}{2(\alpha_2-1)}-\frac{\alpha_1}{2(\alpha_1-1)}}.e^{B_2x^{\frac{\alpha_2}{\alpha_2-1}}-B_1x^{\frac{\alpha_1}{\alpha_1-1}}}+1\right).$

But $1<\alpha_1<\alpha_2\ and\ so\ \frac{\alpha_1}{2(\alpha_1-1)}>\frac{\alpha_2}{2(\alpha_2-1)}$, so, $x^{\frac{\alpha_2}{2(\alpha_2-1)}-\frac{\alpha_1}{2(\alpha_1-1)}}\to 0$ as $x\to$,

and $e^{B_2x^{\frac{\alpha_2}{\alpha_2-1}}-B_1x^{\frac{\alpha_1}{\alpha_1-1}}}\to 0$ as $x\to$. We obtain

$1-F(x)\ \sim\ qA_2x^{\frac{-\alpha_2}{2(\alpha_2-1)}}\,exp\left\{-B_2x^{\frac{\alpha_2}{\alpha_2-1}}\right\}.$

We now consider the asymptotic behavior of the tail $1-F(x)$, as $x\to$. $For\ x>0$, we have

$$\lim_{t\to}\frac{1-F(t+xg(t))}{1-F(t)}=\lim_{t\to}\frac{qA_2(t+xg(t))^{\frac{-\alpha_2}{2(\alpha_2-1)}}.e^{-B_2(t+xg(t))^{\frac{\alpha_2}{\alpha_2-1}}}}{qA_2\,.\,t^{\frac{-\alpha_2}{2(\alpha_2-1)}}.e^{-B_2t^{\frac{\alpha_2}{\alpha_2-1}}}}$$

$$=\lim_{t\to}\ \left(1+x\frac{g(t)}{t}\right)^{\frac{-\alpha_2}{2(\alpha_2-1)}}.e^{-B_2t^{\frac{\alpha_2}{\alpha_2-1}}\left\{\left(1+x\frac{g(t)}{t}\right)^{\frac{\alpha_2}{(\alpha_2-1)}}-1\right\}}. \tag{2.31}$$

Now: $e^{-B_2t^{\frac{\alpha_2}{\alpha_2-1}}\left\{\left(1+x\frac{g(t)}{t}\right)^{\frac{\alpha_2}{(\alpha_2-1)}}-1\right\}}\sim\ e^{-x}\Leftrightarrow\ B_2t^{\frac{\alpha_2}{\alpha_2-1}}\left\{1+x.\frac{\alpha_2}{(\alpha_2-1)}.\frac{g(t)}{t}-1\right\}\ \sim\ x$

$$\Leftrightarrow g(t)\sim\frac{1}{B_2}.\frac{\alpha_2-1}{\alpha_2}t^{\frac{-1}{\alpha_2-1}}.$$

Then $\left(1+x\frac{g(t)}{t}\right)^{\frac{\alpha_2}{(\alpha_2-1)}}\ \sim\left\{1+x.\frac{\alpha_2}{(\alpha_2-1)}.\frac{g(t)}{t}\right\}$, when $x\frac{g(t)}{t}$ is near zero.

Now, take $g(t)=\frac{1}{B_2}.\frac{\alpha_2-1}{\alpha_2}t^{\frac{-1}{\alpha_2-1}}$ and substitute $g(t)$ into (2.31). We obtain

$$\lim_{t\to}\ \frac{1-F(t+xg(t))}{1-F(t)}=\lim_{t\to}\ \left(1+x\frac{1}{B_2}.\frac{\alpha_2-1}{\alpha_2}t^{\frac{-\alpha_2}{\alpha_2-1}}\right)^{\frac{-\alpha_2}{2(\alpha_2-1)}}.e^{-B_2t^{\frac{\alpha_2}{\alpha_2-1}}\left\{\left(1+x\frac{1}{B_2}.\frac{\alpha_2-1}{\alpha_2}t^{\frac{-\alpha_2}{\alpha_2-1}}\right)^{\frac{\alpha_2}{(\alpha_2-1)}}-1\right\}},$$

$$=\ \lim_{t\to}\ e^{-B_2t^{\frac{\alpha_2}{\alpha_2-1}}\left\{\left(1+x\frac{1}{B_2}.\frac{\alpha_2-1}{\alpha_2}.\frac{\alpha_2}{\alpha_2-1}t^{\frac{-\alpha_2}{\alpha_2-1}}\right)+o\left(t^{\frac{-\alpha_2}{\alpha_2-1}}\right)-1\right\}}\ =\lim_{t\to}\ e^{-B_2t^{\frac{\alpha_2}{\alpha_2-1}}\left\{\left(x\frac{1}{B_2}.t^{\frac{-\alpha_2}{\alpha_2-1}}\right)+o\left(t^{\frac{-\alpha_2}{\alpha_2-1}}\right)\right\}},$$

$$=\lim_{t\to}\ e^{-x+o(1)}=e^{-x},\text{as } t\to\ ,$$

since $\lim_{t\to} \left(1 + x\frac{1}{B_2}.\frac{\alpha_2-1}{\alpha_2}t^{\frac{-\alpha_2}{\alpha_2-1}}\right)^{\frac{-\alpha_2}{2(\alpha_2-1)}} \to 1$ as $t \to$.

Therefore, the distribution function $F(x)$ belongs to the domain of attraction of the function $G_0(x)$, and we have the type (I) of extreme value distribution, i.e. there exist constants a_n and b_n, such that the following equality holds true:

$$\lim_{n\to} P\left\{M_n^* \leq \frac{x}{a_n^*} + b_n^*\right\} = \exp(-e^{-x}).$$

We now determine the constants a_n^* and b_n^* .

Let us first determine the constant u_n, such that $1 - F(u_n) \sim \frac{1}{n}e^{-x}$ as $n \to$, i.e.

$$1 - pF_1(u_n) - qF_2(u_n) \sim \frac{1}{n}e^{-x} \quad \text{as } n \to \quad ,$$

$$qA_2u_n^{\frac{-\alpha_2}{2(\alpha_2-1)}}.e^{-B_2u_n^{\frac{\alpha_2}{\alpha_2-1}}} \sim \frac{1}{n}e^{-x} \quad \text{as } n \to \quad . \tag{2.32}$$

Asymptotic relation (2.32) can be written in the following way:

$\frac{1}{n}e^{-x}.\frac{1}{qA_2}.u_n^{\frac{\alpha_2}{2(\alpha_2-1)}}.e^{B_2u_n^{\frac{\alpha_2}{\alpha_2-1}}} \to 1$, by taking logarithms we get

$$-\ln n - x - \ln q - \ln A_2 + \frac{\alpha_2}{2(\alpha_2-1)}\ln u_n + B_2u_n^{\frac{\alpha_2}{\alpha_2-1}} \to 0. \tag{2.33}$$

It follows from (2.33) that $\frac{B_2u_n^{\frac{\alpha_2}{\alpha_2-1}}}{\ln n} \to 1$, as $n \to$, and

$$\ln u_n = \frac{\alpha_2-1}{\alpha_2}(\ln\ln n - \ln B_2) + o(1) . \tag{2.34}$$

The relation (2.33) can also be written in the form

$$B_2u_n^{\frac{\alpha_2}{\alpha_2-1}} = x + \ln n + \ln qA_2 - \frac{\alpha_2}{2(\alpha_2-1)}\ln u_n + o(1). \tag{2.35}$$

Now we substitute the value of $\ln u_n$ from (2.34) into (2.35). We obtain

$$B_2u_n^{\frac{\alpha_2}{\alpha_2-1}} = x + \ln n + \ln qA_2\sqrt{B_2} - \frac{1}{2}\ln\ln n + o(1),$$

$$u_n = (B_2^{-1}\ln n)^{\frac{\alpha_2-1}{\alpha_2}}\left\{1+\frac{x+\ln qA_2\sqrt{B_2}-\frac{1}{2}\ln\ln n}{\ln n}+o\left(\frac{1}{\ln n}\right)\right\}^{\frac{\alpha_2-1}{\alpha_2}},$$

$$u_n = \frac{B_2^{\frac{1-\alpha_2}{\alpha_2}}}{\ln n^{\frac{1-\alpha_2}{\alpha_2}}}\left\{1+\frac{\alpha_2-1}{\alpha_2}\left(\frac{x}{\ln n}+\frac{\ln qA_2\sqrt{B_2}}{\ln n}-\frac{\ln\ln n}{2\ln n}\right)\right\}+o\left(\frac{1}{\ln n}\right),$$

$$u_n \sim \frac{\alpha_2-1}{\alpha_2}.\frac{B_2^{\frac{1-\alpha_2}{\alpha_2}}}{\ln n^{\frac{1}{\alpha_2}}}x+ B_2^{\frac{1-\alpha_2}{\alpha_2}}\left\{\frac{1}{\ln n^{\frac{1-\alpha_2}{\alpha_2}}}+\frac{\alpha_2-1}{\alpha_2}\left(\frac{2\ln qA_2\sqrt{B_2}-\ln\ln n}{2\ln n^{1/\alpha_2}}\right)\right\} .$$

Since $u_n \sim \frac{x}{a_n^*}+b_n^*$, as $n\to$ $, we\ have$

$$a_n^* = \frac{\alpha_2}{\alpha_2-1}\,\frac{\ln n^{1/\alpha_2}}{B_2^{\frac{1-\alpha_2}{\alpha_2}}} \quad\text{and}\quad b_n^* = B_2^{\frac{1-\alpha_2}{\alpha_2}}\left\{\frac{1}{\ln n^{\frac{1-\alpha_2}{\alpha_2}}}+\frac{\alpha_2-1}{\alpha_2}\left(\frac{2\ln qA_2\sqrt{B_2}-\ln\ln n}{2\ln n^{1/\alpha_2}}\right)\right\},$$

where $A_2 = \frac{1}{\sqrt{2\pi\alpha_2(\alpha_2-1)}}\left(\alpha_2\widehat{\sigma_{\alpha_2}}\right)^{\frac{\alpha_2}{2(\alpha_2-1)}}$ and $B_2 = (\alpha_2-1)\left(\alpha_2\widehat{\sigma_{\alpha_2}}\right)^{\frac{-\alpha_2}{\alpha_2-1}}$. □

Remark 2.7.4. If we consider $X_n \sim \begin{cases} S_{\alpha_1}(\sigma_1,-1,0), & with\ probability\ p,\\ S_{\alpha_2}(\sigma_2,-1,0), & with\ probability\ q,\end{cases}$

instead of (2.30) the statement of Theorem still holds with $\widehat{\sigma_{\alpha_2}} = \sigma_2\left(\cos\frac{\pi}{2}(2-\alpha_2)\right)^{\frac{-1}{\alpha_2}}$.

THEOREM 2.7.5. *Let* (X_n) *be a sequence of independent random variables such that*

$$X_n \sim \begin{cases} S_\alpha(\sigma_1,-1,0), & with\ probability\ p,\\ S_\alpha(\sigma_2,-1,0), & with\ probability\ q,\end{cases} \quad for\ all\ n$$

where $p,q>0$, $\sigma_1>\sigma_2$ *and* $X\sim\ S_\alpha(\sigma,-1,0)$ *denotes the stable distribution with the tail behavior given by formula (2.27).*

Let $M_n^{**} = max_{1\le j\le n}X_j$. *Then the limiting distribution of* M_n^{**} *is given by*

$P\left\{M_n^{**} \le \frac{x}{a_n^{**}}+b_n^{**}\right\} \to exp(-e^{-x})$, $as\ n\to$ *, where the normalizing constants are*

$$a_n^{**} = \frac{\alpha}{\alpha-1} \frac{\ln n^{1/\alpha}}{B_3^{\frac{1-\alpha}{\alpha}}} \quad and \quad b_n^{**} = B_3^{\frac{1-\alpha}{\alpha}} \left\{ \frac{1}{\ln n^{\frac{1-\alpha}{\alpha}}} + \frac{\alpha-1}{\alpha} \left(\frac{2 \ln pA_3\sqrt{B_3} - \ln\ln n}{2 \ln n^{1/\alpha}} \right) \right\},$$

where $A_3 = \frac{1}{\sqrt{2\pi\alpha(\alpha-1)}} \left(\alpha\widehat{\sigma_{\alpha,1}}\right)^{\frac{\alpha}{2(\alpha-1)}}$, $B_3 = (\alpha-1)\left(\alpha\widehat{\sigma_{\alpha,1}}\right)^{\frac{-\alpha}{\alpha-1}}$ *and*

$\widehat{\sigma_{\alpha,1}} = \sigma_1 \left(\cos\frac{\pi}{2}(2-\alpha)\right)^{-1/\alpha}$.

In this case the proof has also two main parts: in the first part we obtain that the distribution function of M_n^{**} belongs to the domain of attraction of Gumbel distribution. After that we determine the normalizing constants.

Proof: Since $\sigma_1 > \sigma_2$ then $\widehat{\sigma_{\alpha,1}} > \widehat{\sigma_{\alpha,2}}$.

As before we compute asymptotic behavior of the tail $1 - F(x)$, as $x \to +$.

Similarly as in the proof of Theorem 2.7.3 by finding the dominant part of the distribution, we obtain

$$1 - F(x) \sim pA_3 x^{\frac{-\alpha}{2(\alpha-1)}} \exp\left\{-B_3 x^{\frac{\alpha}{\alpha-1}}\right\},$$

where $A_3 = \frac{1}{\sqrt{2\pi\alpha(\alpha-1)}} \left(\alpha\widehat{\sigma_{\alpha,1}}\right)^{\frac{\alpha}{2(\alpha-1)}} > 0$, $B_3 = (\alpha-1)\left(\alpha\widehat{\sigma_{\alpha,1}}\right)^{\frac{-\alpha}{\alpha-1}} > 0$ *and*

$\widehat{\sigma_{\alpha,1}} = \sigma_1 \left(\cos\frac{\pi}{2}(2-\alpha)\right)^{-1/\alpha}$.

We now consider the asymptotic behavior of the tail $1 - F(x)$, as $x \to$. For $x > 0$, we have

$$\lim_{t\to} \frac{1 - F(t + xg(t))}{1 - F(t)} = \lim_{t\to} \frac{pA_3 (t + xg(t))^{\frac{-\alpha}{2(\alpha-1)}} . e^{-B_3 (t + xg(t))^{\frac{\alpha}{\alpha-1}}}}{pA_3 . t^{\frac{-\alpha}{2(\alpha-1)}} . e^{-B_3 t^{\frac{\alpha}{\alpha-1}}}}$$

$$= \lim_{t\to} \left(1 + x\frac{g(t)}{t}\right)^{\frac{-\alpha}{2(\alpha-1)}} . e^{-B_3 t^{\frac{\alpha}{\alpha-1}} \left\{ \left(1 + x\frac{g(t)}{t}\right)^{\frac{\alpha}{\alpha-1}} - 1 \right\}}. \tag{2.36}$$

Now: $e^{-B_3 t^{\frac{\alpha}{\alpha-1}} \left\{ \left(1 + x\frac{g(t)}{t}\right)^{\frac{\alpha}{\alpha-1}} - 1 \right\}} \sim e^{-x}$

$\Leftrightarrow g(t) \sim \frac{1}{B_3} . \frac{\alpha - 1}{\alpha} t^{\frac{-1}{\alpha-1}}$.

Now, take $g(t) = \frac{1}{B_3} . \frac{\alpha-1}{\alpha} t^{\frac{-1}{\alpha-1}}$ and substitute $g(t)$ into(2.36). We obtain

$$\lim_{t\to} \frac{1-F(t+xg(t))}{1-F(t)} = \lim_{t\to} \left(1 + x\frac{1}{B_3} . \frac{\alpha-1}{\alpha} t^{\frac{-\alpha}{\alpha-1}}\right)^{\frac{-\alpha}{2(\alpha-1)}} . e^{-B_3 t^{\frac{\alpha}{\alpha-1}}\left\{\left(1+x\frac{1}{B_3}\frac{\alpha-1}{\alpha}t^{\frac{-\alpha}{\alpha-1}}\right)^{\frac{\alpha}{\alpha-1}}-1\right\}},$$

$$= \lim_{t\to} e^{-B_3 t^{\frac{\alpha}{\alpha-1}}\left\{\left(x\frac{1}{B_3}.t^{\frac{-\alpha}{\alpha-1}}\right)+o\left(t^{\frac{-\alpha}{\alpha-1}}\right)\right\}},$$

$$= \lim_{t\to} e^{-x+O(1)} = e^{-x}, \text{ as } t \to \quad ,$$

since $\lim_{t\to} \left(1 + x\frac{1}{B_3} . \frac{\alpha-1}{\alpha} t^{\frac{-\alpha}{\alpha-1}}\right)^{\frac{-\alpha}{2(\alpha-1)}} \to 1$ as $t \to$.

Hence, the distribution function $F(x)$ belongs to the domain of attraction of the function $G_0(x)$, and we have the type (I) of extreme value distribution, i.e. there exist constants a_n and b_n, such that the following equality holds true:

$$\lim_{n\to} P\left\{M_n^{**} \le \frac{x}{a_n^{**}} + b_n^{**}\right\} = \exp(-e^{-x}).$$

We now compute a_n^{**} and b_n^{**} .

We first find u_n , such that $1 - F(u_n) \sim \frac{1}{n} e^{-x}$ as $n \to$, i.e.

$$pA_3 u_n^{\frac{-\alpha}{2(\alpha-1)}} . e^{-B_3 u_n^{\frac{\alpha}{\alpha-1}}} \sim \frac{1}{n} e^{-x} \text{, as } n \to \quad . \tag{2.37}$$

Asymptotic relation (2.37) can be transformed in the following way:

$\frac{1}{n} e^{-x} . \frac{1}{pA_3} . u_n^{\frac{\alpha}{2(\alpha-1)}} . e^{B_3 u_n^{\frac{\alpha}{\alpha-1}}} \to 1$, by taking logarithms, we obtain

$$B_3 u_n^{\frac{\alpha}{\alpha-1}} = x + \ln n + \ln pA_3 - \frac{\alpha}{2(\alpha-1)} \ln u_n + o(1) .$$

$$B_3 u_n^{\frac{\alpha}{\alpha-1}} = x + \ln n + \ln pA_3\sqrt{B_3} - \frac{1}{2} \ln \ln n + o(1),$$

$$u_n = (B_3^{-1} \ln n)^{\frac{\alpha-1}{\alpha}} \left\{1 + \frac{x + \ln pA_3\sqrt{B_3} - \frac{1}{2}\ln\ln n}{\ln n} + o\left(\frac{1}{\ln n}\right)\right\}^{\frac{\alpha-1}{\alpha}},$$

$$u_n \sim \frac{\alpha-1}{\alpha} . \frac{B_3^{\frac{1-\alpha}{\alpha}}}{\ln n^{\frac{1}{\alpha}}} x + B_3^{\frac{1-\alpha}{\alpha}} \left\{\frac{1}{\ln n^{\frac{1-\alpha}{\alpha}}} + \frac{\alpha-1}{\alpha}\left(\frac{2\ln pA_3\sqrt{B_3} - \ln\ln n}{2\ln n^{1/\alpha}}\right)\right\} .$$

Since $u_n \sim \frac{x}{a_n^{**}} + b_n^{**}$, as $n \to$, we have

$$a_n^{**} = \frac{\alpha}{\alpha-1} \frac{\ln n^{1/\alpha}}{B_3^{\frac{1-\alpha}{\alpha}}} \text{ and } b_n^{**} = B_3^{\frac{1-\alpha}{\alpha}} \left\{ \frac{1}{\ln n^{\frac{1-\alpha}{\alpha}}} + \frac{\alpha-1}{\alpha} \left(\frac{2 \ln p A_3 \sqrt{B_3} - \ln \ln n}{2 \ln n^{1/\alpha}} \right) \right\},$$

where $A_3 = \frac{1}{\sqrt{2\pi\alpha(\alpha-1)}} \left(\alpha\widehat{\sigma_{\alpha,1}}\right)^{\frac{\alpha}{2(\alpha-1)}}$ and $B_3 = (\alpha-1)\left(\alpha\widehat{\sigma_{\alpha,1}}\right)^{\frac{-\alpha}{\alpha-1}}$. □

2.8. Mixture of an infinite sequence of independent normally distributed variables

Also, in this chapter, we study distribution of extreme values of a mixture of an infinite sequence of independent normally distributed variables with the same mean and an increasing sequence of standard deviations [7]. This paper also extends the result of Mladenović, P., [20], where extreme values of mixture of two independent normally distributed variables were studied, to the case of a mixture of an infinite sequence of such variables.

We will show that the common distribution function of a mixture of an infinite sequence of independent normally distributed variables belongs to the domain of attraction of type (I).

Thus, limiting distribution of the maximum of the mixture is given by

$$P(a_n(M_n - b_n) \le x) \to e^{-e^{-x}},$$

with the normalizing constants a_n *and* b_n also computed in this paper.

Our result will make use of two Theorems from [17], the first Theorem (1.5.2) in chapter 1, which is Theorem 1.6.2 from [17], enables us to determine the domain of attraction and its type for our common distribution function.

The second Theorem (1.5.12) in chapter 1, which is Theorem 1.5.1 from [17], enables us to find the normalizing constants a_n and b_n in (1.2).

In this paper we consider an infinite sequence of normally distributed random variables, $Z_k \sim N(\mu_k, \sigma_k^2)$, such that

$$\mu_k = \mu_0, \qquad k \ge 1$$

and $\quad 0 < \sigma_1 < \sigma_2 < \cdots < \sigma_k \to \sigma_0 \text{ as } k \to \infty.$

The distribution function of variable Z_k is

$F_k(x) = P(Z_k \le x) = \Phi\left(\frac{x-\mu_k}{\sigma_k}\right)$, where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt$.

Then a random variable X which is a mixture of an infinite sequence of variables $Z_1, Z_2, \dots, Z_k, \dots$ with probability $p_1, p_2, \dots, p_k$, $\sum_{k=1}^{\infty} p_k = 1$, has the distribution function given by

$$F(x) = \sum_{k=1}^{\infty} p_k \, \Phi\left(\frac{x-\mu_k}{\sigma_k}\right). \tag{2.38}$$

Main results

We are now ready to determine the type of the domain of attraction and corresponding normalizing constants for our mixture.

Theorem 2.8.1 *Let* (X_n) *be a sequence of independent random variables with common distribution* $F(x)$ *defined by (2.38), then F belongs to the domain of attraction of Gumbel extreme value distribution, i.e., there exist norming constants* $a_n > 0$ *and* b_n *such that*

$lim_{n\to\infty} P\left(M_n \le \frac{x}{a_n} + b_n\right) = lim_{n\to\infty} F\left(\frac{x}{a_n} + b_n\right) = exp(-e^{-x})$, where $M_n = max\{X_1, X_2, \dots, X_n\}$.

Proof: Let $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x} \varphi(t)dt$.

We shall use the following asymptotic relation

$1 - \Phi(x) = \frac{1}{x}\varphi(x)\big(1 + R(x)\big)$ *where* $R(x) \to 0$ *as* $x \to \infty$.

Suppose $X_k \in N(\mu_k, \sigma_k^2)$ *for* $k \in N$. Then

$$1 - F(t) = \sum_{k=1}^{\infty} p_k \left(\varphi\left(\frac{t-\mu_k}{\sigma_k}\right)\frac{\sigma_k}{t-\mu_k}\right)\left(1 + R\left(\frac{t-\mu_k}{\sigma_k}\right)\right).$$

Let $R_1(t) = \max_{1\le k<\infty} \left|R\left(\frac{t-\mu_k}{\sigma_k}\right)\right|$.

Since $\mu_k = \mu_0$ *and* $\sigma_1 \le \sigma_k$, *for* $t > 0$ *we have* $\frac{t-\mu_k}{\sigma_k} \ge \frac{t-\mu_0}{\sigma_1}$.

Therefore, since $R(x) \to 0$ *as* $x \to \infty$, $R_1(t) \to 0$ *as* $t \to \infty$.

Hence, $\left|1 - F(t) - \sum_{k=1}^{\infty} p_k \left(\varphi\left(\frac{t-\mu_k}{\sigma_k}\right)\frac{\sigma_k}{t-\mu_k}\right)\right| < R_1(t) \sum_{k=1}^{\infty} p_k \left(\varphi\left(\frac{t-\mu_k}{\sigma_k}\right)\frac{\sigma_k}{t-\mu_k}\right) \to 0$, *as* $t \to \infty$.

Therefore

$1-F(t) = \mathrm{o}(1) + \sum_{k=1}^{\infty} p_k \left(\varphi\left(\frac{t-\mu_k}{\sigma_k}\right)\frac{\sigma_k}{t-\mu_k}\right),$

$$1-F(t)+\mathrm{o}(1) = \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty} p_k \frac{\sigma_k}{t-\mu_k} e^{-\frac{1}{2}\left(\frac{t-\mu_k}{\sigma_k}\right)^2},$$

$$1-F(t)+\mathrm{o}(1) = \frac{1}{\sqrt{2\pi}}\frac{\sigma_0}{t-\mu_0} e^{-\frac{1}{2}\left(\frac{t-\mu_0}{\sigma_0}\right)^2}\sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0}\frac{t-\mu_0}{t-\mu_k} e^{-\frac{1}{2}\Delta_k(t)},$$

where $\Delta_k(t) = \left(\frac{t-\mu_k}{\sigma_k}\right)^2 - \left(\frac{t-\mu_0}{\sigma_0}\right)^2,$

$$1-F(t)+\mathrm{o}(1) = \frac{1}{\sqrt{2\pi}}\frac{\sigma_0}{t-\mu_0} e^{-\frac{1}{2}\left(\frac{t-\mu_0}{\sigma_0}\right)^2} p(t),$$

where $p(t) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0}\frac{t-\mu_0}{t-\mu_k} e^{-\frac{1}{2}\Delta_k(t)}$.

Let $g(t) = \frac{\sigma_0^2}{t-\mu_0}$.

We have to prove that $\frac{p(t+xg(t))}{p(t)} \to 1 \ as \ t \to \infty$.

Note that p is an analytic function. Using the Taylor expansion with the Lagrange form of the reminder, we get $p(t+h) = p(t) + p'(\xi)h$, where $t \le \xi \le t+h$,

$$\frac{p(t+h)}{p(t)} = \frac{p(t)+p'(\xi)h}{p(t)} = 1 + \frac{p'(\xi)}{p(t)}h + o(h).$$

Put $h = xg(t) = \frac{x\sigma_0^2}{t-\mu_0}$.

Note that $h \to 0 \ as \ t \to \infty$.

We have $ht_1 = x\sigma_0^2 \ where \ t_1 = t-\mu_0$.

Since or assumption is that $\mu_k = \mu_0$, we have

$\Delta_k(t) = \left(\frac{t-\mu_0}{\sigma_k}\right)^2 - \left(\frac{t-\mu_0}{\sigma_0}\right)^2 = t_1^2\left(\frac{1}{\sigma_k^2} - \frac{1}{\sigma_0^2}\right) = \delta_k t_1^2, \ where \ \delta_k = \frac{1}{\sigma_k^2} - \frac{1}{\sigma_0^2}.$

Hence, $p(t) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} e^{-\frac{1}{2}\delta_k t_1^2}$.

We now compute, using for instance the Dominated Convergence Theorem to justify term by term differentiation:

$$p'(t) = -\sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \delta_k t_1 e^{-\frac{1}{2}\delta_k t_1^2},$$

$$-\frac{p'(t)}{t_1} = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \delta_k e^{-\frac{1}{2}\delta_k t_1^2}.$$

Note that $\delta_k \to 0 \; as \; k \to \infty$.

Note also that $|p'(t)|$ is a decreasing function of t, and hence $\frac{p'(\xi)}{p(t)} h \to 0 \; if \; \frac{p'(t)}{p(t)} h \to 0$.

Since $h = xg(t) = \frac{x\sigma_0^2}{t_1}$, if x is kept constant, the condition $\frac{p'(t)}{t_1 p(t)} \to 0 \; as \; t \to \infty$ will imply

$\frac{p'(t)}{p(t)} h \to 0 \; as \; t \to \infty$.

We have:

$$\frac{p'(t)}{p(t)t_1} = -\frac{\sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \delta_k e^{-\frac{1}{2}\delta_k t_1^2}}{\sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \delta_k e^{-\frac{1}{2}\delta_k t_1^2}} \to 0 \; as \; t_1 \to \infty,$$

and therefore

$\frac{p(t+xg(t))}{p(t)} = 1 + o(1) \; as \; t \to \infty$.

Note also that if x is kept constant, $\frac{t_1}{t_1 + xg(t)} = 1 + o(1) \; as \; t \to \infty \; too$.

We have in fact also proven that $\frac{p\left(t + o\left(\frac{1}{t}\right)\right)}{p(t)} = 1 + o(1) \; as \; t \to \infty$. (2.39)

We now consider asymptotic behavior as $x \to \infty$.

For $x > 0$ we have, as $t \to \infty$

$$\frac{1 - F(t + xg(t))}{1 - F(t)} = e^{\frac{t_1^2 - \left(t_1 + xg(t)\right)^2}{2\sigma_0^2}} \frac{t_1}{t_1 + xg(t)} \frac{p\left(t + xg(t)\right)}{p(t)} \left(1 + o(1)\right)$$
$$= e^{-\frac{x^2 g(t)^2}{2\sigma_0^2}} e^{-\frac{xg(t)t_1}{\sigma_0^2}} \left(1 + o(1)\right).$$

Recall that $g(t) = \frac{\sigma_0^2}{t_1}$, and hence

$$\frac{1-F(t+xg(t))}{1-F(t)} = e^{-\frac{x^2 g(t)^2}{2\sigma_0^2}} e^{-x}\big(1+o(1)\big) \to e^{-x}\ as\ t \to \infty.$$

We conclude that the distribution function $F(x)$ belongs to the domain of attraction of the function $G_0(x)$, and we have the type I of extreme value distribution. □

Now we proceed to find the normalizing constants $a_n^*\ and\ b_n^*$.

Theorem 2.8.2. *Limiting distribution of the maximum of the mixture, in notation of the previous theorem, is given by*

$P\{a_n^*(M_n - b_n^*) \le x\} \to e^{-e^{-x}}$, *where*

$$a_n^* = \frac{\sqrt{2\,ln\,n}}{\sigma_0}, \qquad b_n^* = \mu_0 + \sigma_0\sqrt{2\,ln\,n} - \frac{\sigma_0}{2\sqrt{2\,ln\,n}}(ln\,ln\,n + ln\,4\pi) - \tau_n,$$

with $\tau_n > 0$ *the smallest positive solution of an equation*

$$\tau = \frac{\sigma_0}{\sqrt{2\,ln\,n}}\left|ln\,p(\mu_0 + \sigma_0\sqrt{2\,ln\,n} - \frac{\sigma_0}{2\sqrt{2\,ln\,n}}(ln\,ln\,n + ln\,4\pi) - \tau)\right|$$

When $2\,ln\,n - \frac{1}{2}(ln\,ln\,n + ln\,4\pi) > |ln\,p(\mu_0)|$, *and 0 otherwise.*

Proof: Let $v_n^{(0)} = \frac{u_n - \mu_0}{\sigma_0}\ and\ v_n^{(k)} = \frac{u_n - \mu_k}{\sigma_k}$, so that

$$F_k(u_n) = \Phi\big(v_n^{(k)}\big).$$

We have (see proof of the previous theorem)

$$1 - \sum_{k=1}^{\infty} p_k\,\Phi\big(v_n^{(k)}\big) \sim \frac{\sum_{k=1}^{\infty} p_k\,\varphi\big(v_n^{(k)}\big)}{v_n^{(k)}} = \frac{1}{\sqrt{2\pi}v_n^{(0)}} e^{-\frac{1}{2}\left(v_n^{(0)}\right)^2} p(u_n)\big(1+o(1)\big),$$

where

$$p(u_n) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0}\frac{u_n - \mu_0}{u_n - \mu_k} e^{-\frac{1}{2}\Delta_k(u_n)},$$

$$\Delta_k(u_n) = \left(\frac{u_n - \mu_k}{\sigma_k}\right)^2 - \left(\frac{u_n - \mu_0}{\sigma_0}\right)^2 = \left(\frac{1}{\sigma_k^2} - \frac{1}{\sigma_0^2}\right)u_n^2 + Au_n + B \to \infty\ as\ n \to \infty.$$

Hence the constant u_n should be determined from the conditions $v_n^{(0)} = \frac{u_n-\mu_0}{\sigma_0}$,

i.e. $u_n = \mu_0 + \sigma_0 v_n^{(0)}$, and

$$p(u_n)\frac{\varphi\left(v_n^{(0)}\right)}{v_n^{(0)}} \sim \frac{1}{n} e^{-x} \; as \; n \to \infty. \tag{2.40}$$

This asymptotic relation can be rewritten as

$$\frac{1}{np(u_n)} e^{-x} \frac{v_n^{(0)}}{\varphi\left(v_n^{(0)}\right)} \to 1,$$

Or, by taking logarithms,

$$-\ln n - \ln\big(p(u_n)\big) - x + \ln\left(v_n^{(0)}\right) - \ln\left(\varphi\left(v_n^{(0)}\right)\right) \to 0,$$

which is equivalent to

$$-\ln n - \ln\big(p(u_n)\big) - x + \ln\left(v_n^{(0)}\right) - \left(-\frac{1}{2}\ln 2\pi - \frac{1}{2}\left(v_n^{(0)}\right)^2\right) \to 0,$$

i.e.

$$-\ln n - \ln\big(p(u_n)\big) - x + \ln\left(v_n^{(0)}\right) + \frac{1}{2}\ln 2\pi + \frac{1}{2}\left(v_n^{(0)}\right)^2 \to 0.$$

Provided that $\frac{\ln(p(u_n))}{\ln n} = o(1)$, we will have $\frac{\left(v_n^{(0)}\right)^2}{2\ln n} \to 1 \; as \; n \to \infty$, and hence, by taking logarithms again

$$2\ln\left(v_n^{(0)}\right) - \ln 2 - \ln\ln n = o(1).$$

Hence

$$\ln\left(v_n^{(0)}\right) = \frac{1}{2}(\ln 2 + \ln\ln n) + o(1). \tag{2.41}$$

Substituting back this expression for $\ln\left(v_n^{(0)}\right)$, we find

$$\frac{1}{2}\left(v_n^{(0)}\right)^2 = x + \ln n + \ln\big(p(u_n)\big) - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n + o(1). \tag{2.42}$$

Conversely, if (2.42) holds and $\frac{\ln(p(u_n))}{\ln n} = o(1)$, the right hand side of (2.42) will be equal to

$\ln n\big(1+o(1)\big)$ and hence, taking logarithm of both sides, we see that (2.41) will also hold, and hence (2.40) holds as well.

We will choose a sequence u_n so that it satisfies $\frac{\ln(p(u_n))}{\ln n} = o(1)$.

For this, it is sufficient that $u_n = o\big(\sqrt{\ln n}\big)\ as\ n \to \infty$.

To check that this is indeed enough, we use the assumption that $\mu_k = \mu_0$.

Hence the formula $p(u_n) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} e^{-\frac{1}{2}\delta_k (u_n-\mu_0)^2}$

holds, and $p(u_n) > \frac{p_k\sigma_k}{\sigma_0} e^{-\frac{1}{2}\delta_k (u_n-\mu_0)^2}$,

i.e. $|\ln p(u_n)| < \frac{1}{2}\delta_k(u_n-\mu_0)^2 - \ln\left(\frac{p_k\sigma_k}{\sigma_0}\right)$.

Now using $u_n = o\big(\sqrt{\ln n}\big)$, we get

$\frac{|\ln p(u_n)|}{\ln n} < \delta_k o(1)$, for every k, and thus $\frac{\ln(p(u_n))}{\ln n} = o(1)$, since $\delta_k \to 0\ as\ k \to \infty$.

Now let us find $v_n^{(0)}$ so that (2.42) holds.

Let $w_n = \mu_0 + \sigma_0\sqrt{2\ln n} - \frac{\sigma_0}{2\sqrt{2\ln n}}(\ln\ln n + \ln 4\pi)$.

Note that $p(\mu_0) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0}$ is the maximum of function p, and for $t > \mu_0$ the unction $p(t)$ is decreasing.

Define τ_n to be 0 when $w_n - \mu_0 \le \frac{\sigma_0}{\sqrt{2\ln n}}\big|\ln\big(p(\mu_0)\big)\big|$, and τ_n to be a solution of an equation

$$\tau = \frac{\sigma_0}{\sqrt{2\ln n}}\big|\ln\big(p(w_n-\tau)\big)\big|, \tag{2.43}$$

otherwise.

The solution to (2.43) exists and is unique when $w_n - \mu_0 > \frac{\sigma_0}{\sqrt{2\ln n}}\big|\ln\big(p(\mu_0)\big)\big|$, since the right hand side of (2.43) is a decreasing function of τ for $0 < \tau < w_n - \mu_0$, and becomes smaller than the left hand side for $\tau = w_n - \mu_0$.

The solution will satisfy $0 < \tau_n < w_n - \mu_0$.

But since $w_n = o\big(\sqrt{\ln n}\big)$ we will have that $p(w_n - \tau_n) = o(\ln n)$ and hence $\tau = o\big(\sqrt{\ln n}\big)$, because of (2.43).

Note that since $w_n \to 0\ as\ n \to \infty$, for n sufficiently large the condition

$w_n - \mu_0 > \frac{\sigma_0}{\sqrt{2\ln n}}\left|\ln\left(p(\mu_0)\right)\right|$ will be satisfied.

We will use the sequence $u_n = w_n - \tau_n + \frac{\sigma_0 x}{\sqrt{2\ln n}}$.

Note that since $\tau_n = o\left(\sqrt{\ln n}\right)$ we have $u_n \sim \sigma_0\sqrt{2\ln n}$.

Taking square roots in (2.42) and using Taylor expansion for $\sqrt{(1+\varepsilon)}$, we see that

$$v_n^{(0)} = \sqrt{2\ln n}\left(1 + \frac{1}{2\ln n}\left(x - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n + \ln\left(p(u_n)\right)\right) + o\left(\frac{1}{\ln n}\right)\right), \tag{2.44}$$

Is needed in order to have (2.40), in addition to $u_n = o\left(\sqrt{\ln n}\right)$.

For n sufficiently large, we will have $\tau_n = \frac{\sigma_0}{\sqrt{2\ln n}}\left|\ln p\left(u_n - \frac{\sigma_0 x}{\sqrt{2\ln n}}\right)\right|$.

Using $u_n \sim \sigma_0\sqrt{2\ln n}$ and (2.39), we get that $\frac{\left|\ln p\left(u_n - \frac{\sigma_0 x}{\sqrt{2\ln n}}\right)\right|}{|\ln u_n|} = 1 + o(1)$ and hence

$\tau_n = \frac{\sigma_0}{\sqrt{2\ln n}}|\ln p(u_n)| + o\left(\frac{1}{\ln n}\right)$.

Therefore (2.44) is equivalent to

$$u_n = \mu_0 + \sigma_0\sqrt{2\ln n}\left(1 + \frac{1}{2\ln n}\left(x - \frac{1}{2}\ln 4\pi - \frac{1}{2}\ln\ln n + \ln\left(p(u_n)\right)\right) + o\left(\frac{1}{\ln n}\right)\right),$$

$$u_n = \mu_0 + \frac{\sigma_0 x}{\sqrt{2\ln n}} + \sigma_0\sqrt{2\ln n} - \frac{\sigma_0}{2\sqrt{2\ln n}}(\ln\ln n + \ln 4\pi) - \tau_n + o\left(\frac{1}{\sqrt{\ln n}}\right),$$

$$u_n = w_n + \frac{\sigma_0 x}{\sqrt{2\ln n}} - \tau_n + o\left(\frac{1}{\sqrt{\ln n}}\right).$$

The last equation is obviously satisfied for n large enough, since $u_n = w_n - \tau_n + \frac{\sigma_0 x}{\sqrt{2\ln n}}$.

Note that for u_n the relation $u_n = o\left(\sqrt{\ln n}\right)$ also holds, since $u_n \sim \sigma_0\sqrt{2\ln n}$, and so the corresponding $v_n^{(0)}$ will satisfy all the equations used in the above calculation.

This gives us the required constants in the expression $u_n = \frac{x}{a_n^*} + b_n^*$:

$$a_n^* = \frac{\sqrt{2\ln n}}{\sigma_0},$$

$b_n^* = \mu_0 + \sigma_0\sqrt{2\ln n} - \frac{\sigma_0}{2\sqrt{2\ln n}}(\ln\ln n + \ln 4\pi) - \tau_n,$

Where τ_n is the solution of equation (2.43) when $w_n - \mu_0 > \frac{\sigma_0}{\sqrt{2\ln n}}\left|\ln\left(p(\mu_0)\right)\right|,$ and 0 otherwise.

2.9. Conclusion and Future Research

In Theorems 2.7.2, 2.7.3 and 2.7.5 we established the exact distribution of extreme values for sequences in the following cases (see statement of the theorems for details):

(i) $X_n \sim \begin{cases} S_\alpha(\sigma_1, 0,0), & \text{with probability p,} \\ S_\alpha(\sigma_2, 0,0), & \text{with probability q.} \end{cases}$ for all n . (Theorem 2.7.2)

(ii) $X_n \sim \begin{cases} S_{\alpha_1}(\sigma, -1,0), & \text{with probability p,} \\ S_{\alpha_2}(\sigma, -1,0), & \text{with probability q.} \end{cases}$ for all n. (Theorem 2.7.3)

(iii) $X_n \sim \begin{cases} S_\alpha(\sigma_1, -1,0), & \text{with probability p,} \\ S_\alpha(\sigma_2, -1,0), & \text{with probability q.} \end{cases}$ for all n . (Theorem 2.7.5)

Our results show which domain of attraction the extreme value distributions belong to. In the first case, we have the domain of attraction of the function $G_1(x)$, while in the second and in the third case, we have the domain of attraction of the function $G_0(x)$.

We also obtained the normalizing constants and we found that they can depend on the first component of the mixture, on the second component or on both of them, namely:

(i) In the first case they depend on both of components of the mixture.
(ii) In the second case they depend on the second component of the mixture.
(iii) In the third case they depend on the first component of the mixture.

Thus, we conclude that in all cases, the heaviest tail dominates the limit.

In Theorems 2.8.1. and 2.8.2, we have shown to which domain of attraction the extreme value distributions of an infinite mixture of normally distributed variables belong to.

In Theorem 2.8.1, we have shown that the domain of attraction is of the function $G_0(x)$.

In Theorem 2.8.2, we have also obtained the normalizing constants.

In determining the normalizing constants we had to solve for the correction factor τ, which was not explicitly given.

The magnitude of τ depends on the rate of growth of the function p, $p(t) = \sum_{k=1}^{\infty} p_k \frac{\sigma_k}{\sigma_0} \frac{t-\mu_0}{t-\mu_k} e^{-\frac{1}{2}\Delta_k(t)}$.

For future research, one might try to find explicit constants under some assumptions about growth of function p.

Furthermore, one can try to extend the results of this thesis by considering extreme values of an infinite sequence of variables with some distribution different from the normal distribution.

References

[1] Balkema, A. and de Haan, L. (1974). Residual life time at great age. *Ann. Probab.*, 2:792-804.

[2] Balkema, A.A, and de Haan, L. (1972). On R.von-Mises' condition for the domain of attraction of exp $(-e^{-x})$. *The Annals of Mathematical Statistics*, Vol.43, No.4, 1352-1354.

[3] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation*. Cambridge University Press, Cambridge.

[4] Coles, Stuart (2001). *An Introduction to Statistical Modeling of Extreme Values,*. Springer-Verlag. ISBN 1-85233-459-2.

[5] Davison, A. C. and Smith, R. L. (1990) Models for Exceedances over High Thresholds. *J. R. Statist. Soc.* B, 52, No. 3, pp. 377-392.

[6] Shneina, E. K. (2013). Distribution of extreme values of the sequences of independent random variables with mixed stable distributions, *Filomat* xx, zzz-zzz.

[7] Shneina, E. K. and Božin, V. (2013). Extreme values of an infinite mixture of normally distributed variables. *Filomat*, 27:5, 909-916.

[8] Embrechts, P., C. Klüppelberg, and T. Mikosch (1997) *Modelling extremal events for insurance and finance*. Berlin: Spring Verlag.

[9] Everitt, B.S. and D.J. Hand (1981). *Finite Mixture Distributions*. Chapman and Hall, London.

[10] Feller, W. (1966). *An Introduction to Probability Theory and Its Applications*.

[11] Fisher, R. and Tippett, L. (1928). Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Proceedings of the Cambridge Philosophical Society*, 24:180-190.

[12] Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une serie aleatoire. *Ann. of Math.*, 44:423-453.

[13] Gnedenko B. V., Kolmogorov A. N. (1954). *Limit distributions for sums of independent random variables.*

[14] htt//:www.wikipedia.com.

[15] Karamata, J. (1930). Sur un mode de croissance réguliére des fonctions. *Mathematica* (Cluj), 4, 38-53.

[16] Karamata, J. (1933). Sur un mode de croissance réguliére. Theoremes fondamentaux. *Bull. Soc. Math. France* 61, 55-62.

[17] Leadbetter, M.R., Lindgreen, G. and Rootzén, H. (1983). *Extremes and related properties of random sequences and processes*. Springer-Verlag.

[18] L. de Haan (1970): *On regular variation and its application to the weak convergence of sample extremes*, volume 32 of Mathematical Centre Tracts. Mathematisch Centrum, Amsterdam.

[19] L. de Haan and Resnick, S. I. (1987). Regular variation of probability densities. *SPA*, 25, 83-93.

[20] Mladenović, P., Extreme values of the sequences of independent random variables with mixed distributions, *MATEMATIČKI VESNIK* , 51 (1999), 29- 37.

[21] Mladenović, P. (2002): *Ekstremne vrednosti slučajnih nizova*, Matematički fakultet, Beograd.

[22] Pickands, J. (1975). Statistical inference using extreme order statistics. *Ann. Statist.*, 3:119-131.

[23] Reiss, R.-D. and Thomas, M. (1997). *Statistical Analysis of Extreme Values*. Birkauser, Basel.

[24] Resnick, S. (1971). Tail equivalence and its applications, *J. Applied Probability*, 8, 136-156.

[25] Resnick, S. I. (1987) *Extreme Values, Regular Variation, and Point Processes*, Springer-Verlag, New York-Berlin Heidelberg.

[26] R. von Mises (1936): La distribution de la plus grande de n valeurs. Rev, Math, Union Interbalcanique 1: 141-160(1936). Reproduced in: Selected papers of Richard von Mises. *American Mathematical Society* (1964) 2:271–294.

[27] Samorodnitsky, G. and Taqqu, M. (1994): *Stable non-Gaussian Random Processes*, *Stochastic Models with infinite variance*, Chapman and Hall, New York.

[28] XTREMES (Version 3.0): Xtremes is a Statistical Software system.

Прилог 1.

Изјава о ауторству

Потписани-а Ehfayed Khalifa A. Shneina

број уписа 2001/2008

Изјављујем

да је докторска дисертација под насловом

"Extreme values in sequences of independent random variables with mixed distributions"

- резултат сопственог истраживачког рада,
- да предложена дисертација у целини ни у деловима није била предложена за добијање било које дипломе према студијским програмима других високошколских установа,
- да су резултати коректно наведени и
- да нисам кршио/ла ауторска права и користио интелектуалну својину других лица.

Потпис докторанда

У Београду, ___________________

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Прилог 2.

Изјава о истоветности штампане и електронске верзије докторског рада

Име и презиме аутора Ehfayed Khalifa A. Shneina

Број уписа 2001/2008

Студијски програм Matematika

Наслов рада "Extreme values in sequences of independent random variables with mixed distributions"

Ментор prof. dr Pavle Mladenović

Потписани Ehfayed Khalifa A. Shneina

изјављујем да је штампана верзија мог докторског рада истоветна електронској верзији коју сам предао/ла за објављивање на порталу **Дигиталног репозиторијума Универзитета у Београду.**

Дозвољавам да се објаве моји лични подаци везани за добијање академског звања доктора наука, као што су име и презиме, година и место рођења и датум одбране рада.

Ови лични подаци могу се објавити на мрежним страницама дигиталне библиотеке, у електронском каталогу и у публикацијама Универзитета у Београду.

Потпис докторанда

У Београду, ________________________

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Прилог 3.

Изјава о коришћењу

Овлашћујем Универзитетску библиотеку „Светозар Марковић“ да у Дигитални репозиторијум Универзитета у Београду унесе моју докторску дисертацију под насловом:

“Extreme values in sequences of independent random variables with mixed distributions”

која је моје ауторско дело.

Дисертацију са свим прилозима предао/ла сам у електронском формату погодном за трајно архивирање.

Моју докторску дисертацију похрањену у Дигитални репозиторијум Универзитета у Београду могу да користе свикоји поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons) за коју сам се одлучио/ла.

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3. Ауторство – некомерцијално – без прераде

4. Ауторство – некомерцијално – делити под истим условима

5. Ауторство – без прераде

6. Ауторство – делити под истим условима

(Молимо да заокружите само једну од шест понуђених лиценци, кратак опис лиценци дат је на полеђини листа).

Потпис докторанда

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4. Ауторство - некомерцијално – делити под истим условима. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, и прераде, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце и ако се прерада дистрибуира под истом или сличном лиценцом. Ова лиценца не дозвољава комерцијалну употребу дела и прерада.

5. Ауторство – без прераде. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, без промена, преобликовања или употребе дела у свом делу, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце. Ова лиценца дозвољава комерцијалну употребу дела.

6. Ауторство - делити под истим условима. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, и прераде, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце и ако се прерада дистрибуира под истом или сличном лиценцом. Ова лиценца дозвољава комерцијалну употребу дела и прерада. Слична је софтверским лиценцама, односно лиценцама отвореног кода.