UNIVERZITET U BEOGRADU
MATEMATIČKI FAKULTET

Qusuay Hatim Eghaar Alqifiary

STABILNOST REŠENJA DIFERENCIJALNIH JEDNAČINA U SMISLU HAJERSA I ULAMA

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Qusuay Hatim Eghaar Alqifiary

HYERS-ULAM STABILITY OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract

This thesis has been written under the supervision of my mentor Prof. dr. Julka Knezević-Miljanović at the University of Belgrade in the academic year 2014-2015. The aim of this study is to investigate Hyers-Ulam stability of some types of differential equations, and to study a generalized Hyers-Ulam stability and as well as a special case of the Hyers-Ulam stability problem, which is called the superstability. Therefore, when there is a differential equation, we answer the three main questions:
1- Does this equation have Hyers -Ulam stability?
2- What are the conditions under which the differential equation has stability ?
3- What is a Hyers-Ulam constant of the differential equation?
The thesis is divided into three chapters. Chapter 1 is divided into 3 sections. In this chapter, we introduce some sufficient conditions under which each solution of the linear differential equation $u''(t) + \big(1 + \psi(t)\big)u(t) = 0$ is bounded. Apart from this we prove the Hyers-Ulam stability of it and the nonlinear differential equations of the form $u''(t) + F(t, u(t)) = 0$, by using the Gronwall lemma and we prove the Hyers-Ulam stability of the second-order linear differential equations with boundary conditions. In addition to that we establish the superstability of linear differential equations of second-order and higher order with continuous coefficients and with constant coefficients, respectively. Chapter 2 is divided into 2 sections. In this chapter, by using the Laplace transform method, we prove that the linear differential equation of the nth-order $y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t)$ has the generalized Hyers-Ulam stability. And we prove also the Hyers-Ulam-Rassias stability of the second-order linear differential equations with initial and boundary conditions, as well as linear differential equations of higher order in the form of $y^{(n)}(x) + \beta(x)y(x) = 0$, with initial conditions. Furthermore, we establish the generalized superstability of differential equations of nth-order with initial conditions and investigate the generalized superstability of differential equations of second-order in the form of $y''(x)+p(x)y'(x)+q(x)y(x) = 0$. Chapter 3 is divided into 2 sections. In this chapter, by applying the fixed point alternative method, we give a necessary and sufficient condition in order that the first order linear

system of differential equations $\dot{z}(t) + A(t)z(t) + B(t) = 0$ has the Hyers-Ulam-Rassias stability and find Hyers-Ulam stability constant under those conditions. In addition to that, we apply this result to a second-order differential equation $\ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t) = 0$. Also, we apply it to differential equations with constant coefficient in the same sense of proofs. And we give a sufficient condition in order that the first order nonlinear system of differential equations has Hyers-Ulam stability and Hyers-Ulam-Rassias stability. In addition, we present the relation between practical stability and Hyers-Ulam stability and also Hyers-Ulam-Rassias stability.

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Podaci o mentoru i članovima komisije:

MENTOR:
redovni profesor dr Julka Knezević-Miljanović
Matematički fakultet,
Univerzitet u Beogradu

ČLANOVI KOMISIJE :
redovni profesor dr Boško Jovanović
Matematički fakultet,
Univerzitet u Beogradu

---

redovni profesor dr Miodrag Spalević
Matematički fakultet,
Univerzitet u Beogradu

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Introduction

This subject dates back to the talk given by the Polish-American mathematician Ulam at the University of Wisconsin in 1940 (see [40]). In that talk, Ulam asked whether an approximate solution of a functional equation must be near an exact solution of that equation. This asking of Ulam is stated as follows:

Theorem 0.0.1. *Let G_1 be a group and let G_2 be a metric group with a metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy).h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?*

One year later, a partial answer to this question was given by D. H. Hyers [5] for additive functions defined on Banach spaces:

Theorem 0.0.2. *Let $f : X_1 \to X_2$ be a function between Banach spaces such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \delta,$$

for some $\delta > 0$ and for all $x, y \in X_1$. Then the limit

$$A(x) = lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in X_1$, and $A : X_1 \to X_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \leq \delta$$

for every $x \in X_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X_1$, then the function A is linear.

This result is called the Hyers-Ulam Stability of additive Cauchy equation $g(x + y) = g(x)+g(y)$. After Hyers's result, many mathematicians have extended Ulam's

problem to other functional equations and generalized Hyers's result in various directions (see [6, 32, 38, 44]).
Ten years after the publication of Hyerss theorem, D. G. Bourgin extended the theorem of Hyers and stated it in his paper [4] without proof. Unfortunately, it seems that this result of Bourgin failed to receive attention from mathematicians at that time. No one has made use of this result for a long time.
In 1978, Rassias [44] introduced a new functional inequality that we call Cauchy-Rassias inequality and succeeded in extending the result of Hyers, by weakening the condition for the Cauchy differences to unbounded map as follows:

Theorem 0.0.3. *Let $f : X_1 \to X_2$ be a function between Banach spaces. If f satisfies the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$, p with $0 \leq p < 1$ and for any $x, y \in X_1$, then there exists a unique additive function $A : X_1 \to X_2$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for each $x \in X_1$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X_1$, then the function A is linear.

The stability phenomenon of this kind is called the Generalized Hyers-Ulam Stability (or Hyers-Ulam-Rassias stability). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation $\varphi(f, y, y', \ldots, y^{(n)}) = 0$ has Hyers-Ulam stability if for given $\varepsilon > 0$ and a function y such that $|\varphi(f, y, y', \ldots, y^{(n)})| \leq \varepsilon$, there exists a solution y_a of the differential equation such that $|y(t) - y_a(t)| \leq K(\varepsilon)$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$. If the preceding statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(t)$ and $\Phi(t)$, where φ, Φ are appropriate functions not depending on y and y_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability (or Hyers-Ulam-Rassias stability).
Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [21, 22]). Thereafter, Alsina and Ger published their paper [3], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$: If a differentiable function $y(t)$ is a solution of the inequality $|y'(t) - y(t)| \leq \varepsilon$ for any $t \in (a, \infty)$, then there exists a constant c such that $|y(t) - ce^t| \leq 3\varepsilon$ for all $t \in (a, \infty)$. Since then, this problem now known as the problem of Hyers-Ulam stability - has been extensively investigated for the algebraic, functional, differential, integral, and operator equations.

Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order [33–37, 42, 43].
Rus investigated the Hyers-Ulam stability of differential and integral equations using the Gronwall lemma and the technique of weakly Picard operators (see [13, 14]). Recently, The results given in [36, 42, 45] have been generalized by Cimpean and Popa [10] and by Popa and Raşa [8, 9] for the linear differential equations.
In 1979, J.Baker, J. Lawrence and F. Zorzitto[16] proved a new type of stability of the exponential equation $f(x+y) = f(x)f(y)$. More precisely, they proved that if a complex-valued mapping f defined on a normed vector space satisfies the inequality $|f(x+y) - f(x)f(y)| \leq \delta$ for some given $\delta > 0$ and for all x, y, then either f is bounded or f is exponential. Such a phenomenon is called the superstability of the exponential equation, which is a special kind of Hyers-Ulam stability. It seems that the results of P. Găvruţa, S. Jung and Y. Li [23] are the earliest one concerning the superstability of differential equations.
This thesis is about stability of some types of differential equations, where we introduce this thesis in three chapters.
Chapter one is titled by Hyers-Ulam stability of Differential Equations. This chapter consists of three sections. *In section 1.1*, we introduce some sufficient conditions under which each solution of the linear differential equation (1.1.2) is bounded. As well as we prove the Hyers-Ulam stability of the linear differential equations of the form (1.1.2). *In section 1.2*, we prove the Hyers-Ulam stability of the nonlinear differential equations of the form (1.2.1) by using the Gronwall lemma. *In section 1.3*, we prove the Hyers-Ulam stability of the second-order linear differential equations with boundary conditions. Furthermore, the superstability of linear differential equations with constant coefficients.
Chapter two is titled by Generlaized Hyers-Ulam stability of Differential equations. This chapter consists of two sections. *In section 2.1*, by using the Laplace transform method, we prove that the linear differential equation of the nth-order

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t)$$

has the generalized Hyers-Ulam stability, where α_k is a scalar, y and f are n times continuously differentiable and of exponential order, respectively. *In section 2.2*, we establish the generalized superstability of differential equations of nth-order with initial conditions and investigate the generalized superstability of differential equations of second-order in the form of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$. In additional, we prove the Hyers-Ulam-Rassias stability of the second-order linear differential equations with initial and boundary conditions as well as linear differential equations of higher order in the form of

$$y^{(n)}(x) + \beta(x)y(x) = 0,$$

with initial conditions

$$y(a) = y^{'}(a) = \cdots = y^{(n-1)}(a) = 0,$$

where $n \in \mathbb{N}^{+}$, $y \in C^{n}[a,b], \beta \in C^{0}[a,b],\ -\infty < a < b < +\infty$.
Chapter three is titled by Hyers-Ulam stability of system of differential equations. This chapter consists of two sections. *In section 3.1*, by applying the fixed point alternative method, we give a necessary and sufficient condition in order that the first order linear system of differential equations $\dot{z}(t) + A(t)z(t) + B(t) = 0$ has the Hyers-Ulam-Rassias stability and find Hyers-Ulam stability constant under those conditions. In addition to that, we apply this result to a second order differential equation $\ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t) = 0$. Also, we apply it to differential equations with constant coefficient in the same sense of proofs. *In section 3.2*, we give a sufficient condition in order that the first order nonlinear system of differential equations has Hyers-Ulam stability and Hyers-Ulam-Rassias stability. In addition, we present the relation between practical stability and Hyers-Ulam stability and also Hyers-Ulam-Rassias stability.

Contents

Chapter 1

Hyers-Ulam Stability of Differential Equations

1.1 Hyers-Ulam Stability of Linear Differential Equations

1.1.1 Boundedness of Solutions of a Second Order Differential Equation

In this subsection, we first introduce and prove a lemma which is a kind of the Gronwall inequality.

Lemma 1.1.1. *[28] Let $u, v : [0, \infty) \to [0, \infty)$ be integrable functions, $c > 0$ be a constant, and let $t_0 \geq 0$ be given. If u satisfies the inequality*

$$u(t) \leq c + \int_{t_0}^{t} u(\tau)v(\tau)d\tau \tag{1.1.1}$$

for all $t \geq t_0$, then

$$u(t) \leq c \exp\left(\int_{t_0}^{t} v(\tau)d\tau\right)$$

for all $t \geq t_0$.

Proof. It follows from (1.1.1) that

$$\frac{u(t)v(t)}{c + \int_{t_0}^{t} u(\tau)v(\tau)d\tau} \leq v(t)$$

for all $t \geq t_0$. Integrating both sides of the last inequality from t_0 to t, we obtain

$$\ln\left(c + \int_{t_0}^{t} u(\tau)v(\tau)d\tau\right) - \ln c \leq \int_{t_0}^{t} v(\tau)d\tau$$

or

$$c + \int_{t_0}^{t} u(\tau)v(\tau)d\tau \leq c\exp\left(\int_{t_0}^{t} v(\tau)d\tau\right)$$

for each $t \geq t_0$, which together with (1.1.1) implies that

$$u(t) \leq c\exp\left(\int_{t_0}^{t} v(\tau)d\tau\right)$$

for all $t \geq t_0$. □

In the following theorem, using Lemma 1.1.1, we investigate sufficient conditions under which every solution of the differential equation

$$u''(t) + \big(1 + \psi(t)\big)u(t) = 0 \tag{1.1.2}$$

is bounded.

Theorem 1.1.2. *[28] Let $\psi : [0, \infty) \to \mathbb{R}$ be a differentiable function. Every solution $u : [0, \infty) \to \mathbb{R}$ of the linear differential equation* (1.1.2) *is bounded provided that $\int_0^\infty |\psi'(t)|dt < \infty$ and $\psi(t) \to 0$ as $t \to \infty$.*

Proof. First, we choose t_0 large enough so that $1 + \psi(t) \geq 1/2$ for all $t \geq t_0$. Multiplying (1.1.2) by $u'(t)$ and integrating it from t_0 to t, we obtain

$$\frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \int_{t_0}^{t} \psi(\tau)u(\tau)u'(\tau)d\tau = c_1$$

for all $t \geq t_0$. Integrating by parts, this yields

$$\frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \frac{1}{2}\psi(t)u(t)^2 - \frac{1}{2}\int_{t_0}^{t} \psi'(\tau)u(\tau)^2 d\tau = c_2 \tag{1.1.3}$$

for any $t \geq t_0$. Then it follows from (1.1.3) that

$$\begin{aligned}\frac{1}{4}u(t)^2 &\leq \frac{1}{2}u'(t)^2 + \frac{1}{2}\cdot\frac{1}{2}u(t)^2 \leq \frac{1}{2}u'(t)^2 + \frac{1}{2}\big(1 + \psi(t)\big)u(t)^2 \\ &= c_2 + \frac{1}{2}\int_{t_0}^{t} \psi'(\tau)u(\tau)^2 d\tau\end{aligned}$$

for all $t \geq t_0$. Thus, it holds that

$$u(t)^2 \leq 4c_2 + 2\int_{t_0}^{t} \psi'(\tau)u(\tau)^2 d\tau \leq 4|c_2| + 2\int_{t_0}^{t} |\psi'(\tau)|u(\tau)^2 d\tau \tag{1.1.4}$$

for any $t \geq t_0$.
In view of Lemma 1.1.1, (1.1.4) and our hypothesis, there exists a constant $M_1 > 0$ such that

$$u(t)^2 \leq 4|c_2| \exp\left(\int_{t_0}^{t} 2|\psi'(\tau)|d\tau\right) < M_1^2$$

for all $t \geq t_0$. On the other hand, since u is continuous, there exists a constant $M_2 > 0$ such that $|u(t)| \leq M_2$ for all $0 \leq t \leq t_0$, which completes the proof. □

Corollary 1.1.3. *[28] Let $\phi : [0, \infty) \to \mathbb{R}$ be a differentiable function satisfying $\phi(t) \to 1$ as $t \to \infty$. Every solution $u : [0, \infty) \to \mathbb{R}$ of the linear differential equation*

$$u''(t) + \phi(t)u(t) = 0 \tag{1.1.5}$$

is bounded provided $\int_0^\infty |\phi'(t)|dt < \infty$.

1.1.2 Hyers-Ulam Stability of Linear Differential Equations of Second Order

Given constants $L > 0$ and $t_0 \geq 0$, let $U(L; t_0)$ denote the set of all functions $u : [t_0, \infty) \to \mathbb{R}$ with the following properties:

(*i*) u is twice continuously differentiable;

(*ii*) $u(t_0) = u'(t_0) = 0$;

(*iii*) $\int_{t_0}^\infty |u'(\tau)|d\tau \leq L$.

We now prove the Hyers-Ulam stability of the linear differential equation (1.1.2) by using the Gronwall inequality.

Theorem 1.1.4. *[28] Given constants $L > 0$ and $t_0 \geq 0$, assume that $\psi : [t_0, \infty) \to \mathbb{R}$ is a differentiable function with $C := \int_{t_0}^\infty |\psi'(\tau)|d\tau < \infty$ and $\lambda := \inf_{t \geq t_0} \psi(t) > -1$. If a function $u \in U(L; t_0)$ satisfies the inequality*

$$\left|u''(t) + \big(1 + \psi(t)\big)u(t)\right| \leq \varepsilon \tag{1.1.6}$$

for all $t \geq t_0$ and for some $\varepsilon \geq 0$, then there exist a solution $u_0 \in U(L;t_0)$ of the differential equation (1.1.2) *and a constant $K > 0$ such that*

$$|u(t) - u_0(t)| \leq K\sqrt{\varepsilon} \tag{1.1.7}$$

for any $t \geq t_0$, where

$$K := \sqrt{\frac{2L}{1+\lambda}}\exp\left(\frac{C}{2(1+\lambda)}\right).$$

Proof. We multiply (1.1.6) with $|u'(t)|$ to get

$$-\varepsilon|u'(t)| \leq u'(t)u''(t) + u(t)u'(t) + \psi(t)u(t)u'(t) \leq \varepsilon|u'(t)|$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii) that

$$-\varepsilon\int_{t_0}^{t}|u'(\tau)|d\tau \leq \frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \int_{t_0}^{t}\psi(\tau)u(\tau)u'(\tau)d\tau \leq \varepsilon\int_{t_0}^{t}|u'(\tau)|d\tau$$

for any $t \geq t_0$.
Integrating by parts and using (iii), we have

$$-\varepsilon L \leq \frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \frac{1}{2}\psi(t)u(t)^2 - \frac{1}{2}\int_{t_0}^{t}\psi'(\tau)u(\tau)^2d\tau \leq \varepsilon L \tag{1.1.8}$$

for all $t \geq t_0$.
Since $1+\lambda > 0$ holds for all $t \geq t_0$, it follows from (1.1.8) that

$$\begin{aligned}\frac{1+\lambda}{2}u(t)^2 &\leq \frac{1}{2}u'(t)^2 + \frac{1+\lambda}{2}u(t)^2 \leq \frac{1}{2}u'(t)^2 + \frac{1}{2}\big(1+\psi(t)\big)u(t)^2\\ &\leq \varepsilon L + \frac{1}{2}\int_{t_0}^{t}\psi'(\tau)u(\tau)^2d\tau\\ &\leq \varepsilon L + \frac{1}{2}\int_{t_0}^{t}|\psi'(\tau)|u(\tau)^2d\tau\end{aligned}$$

or

$$u(t)^2 \leq \frac{2L\varepsilon}{1+\lambda} + \frac{1}{1+\lambda}\int_{t_0}^{t}|\psi'(\tau)|u(\tau)^2d\tau$$

for any $t \geq t_0$.
Applying Lemma 1.1.1, we obtain

$$u(t)^2 \leq \frac{2L\varepsilon}{1+\lambda}\exp\left(\frac{1}{1+\lambda}\int_{t_0}^{t}|\psi'(\tau)|d\tau\right) \leq \frac{2L\varepsilon}{1+\lambda}\exp\left(\frac{C}{1+\lambda}\right)$$

for all $t \geq t_0$. Hence, it holds that

$$|u(t)| \leq \exp\left(\frac{C}{2(1+\lambda)}\right)\sqrt{\frac{2L\varepsilon}{1+\lambda}}$$

for any $t \geq t_0$. Obviously, $u_0(t) \equiv 0$ satisfies the equation (1.1.2) and the conditions (i), (ii), and (iii) such that

$$|u(t) - u_0(t)| \leq K\sqrt{\varepsilon}$$

for all $t \geq t_0$, where $K = \sqrt{\frac{2L}{1+\lambda}}\exp\left(\frac{C}{2(1+\lambda)}\right)$. □

If we set $\phi(t) := 1+\psi(t)$, then the following corollary is an immediate consequence of Theorem 1.1.4.

Corollary 1.1.5. *[28] Given constants $L > 0$ and $t_0 \geq 0$, assume that $\phi : [t_0, \infty) \to \mathbb{R}$ is a differentiable function with $C := \int_{t_0}^{\infty} |\phi'(\tau)| d\tau < \infty$ and $\lambda := \inf_{t \geq t_0} \phi(t) > 0$. If a function $u \in U(L; t_0)$ satisfies the inequality*

$$\left|u''(t) + \phi(t)u(t)\right| \leq \varepsilon$$

for all $t \geq t_0$ and for some $\varepsilon \geq 0$, then there exist a solution $u_0 \in U(L; t_0)$ of the differential equation (1.1.5) and a constant $K > 0$ such that

$$|u(t) - u_0(t)| \leq K\sqrt{\varepsilon}$$

for any $t \geq t_0$, where $K := \exp\left(\frac{C}{2\lambda}\right)\sqrt{\frac{2L}{\lambda}}$.

Example 1.1.1. *[28] Let $\phi : [0, \infty) \to \mathbb{R}$ be a constant function defined by $\phi(t) := a$ for all $t \geq 0$ and for a constant $a > 0$. Then, we have $C = \int_0^{\infty} |\phi'(\tau)| d\tau = 0$ and $\lambda = \inf_{t \geq 0} \phi(t) = a$. Assume that a twice continuously differentiable function $u : [0, \infty) \to \mathbb{R}$ satisfies $u(0) = u'(0) = 0$, $\int_0^{\infty} |u'(\tau)| d\tau \leq L$, and*

$$\left|u''(t) + \phi(t)u(t)\right| = \left|u''(t) + au(t)\right| \leq \varepsilon$$

for all $t \geq 0$ and for some $\varepsilon \geq 0$ and $L > 0$. According to Corollary 1.1.5, there exists a solution $u_0 : [0, \infty) \to \mathbb{R}$ of the differential equation, $y''(t) + ay(t) = 0$, such that

$$|u(t) - u_0(t)| \leq \sqrt{\frac{2L}{a}\varepsilon}$$

for any $t \geq 0$.
Indeed, if we define a function $u : [0, \infty) \to \mathbb{R}$ *by*

$$u(t) := \frac{\alpha}{(t+1)^2} \cos \sqrt{a}t + \frac{2\alpha}{\sqrt{a}(t+1)^2} \sin \sqrt{a}t - \alpha,$$

where we set $\alpha = \frac{\sqrt{a}}{a+\sqrt{a}+2} L$, *then* u *satisfies the conditions stated in the first part of this example, as we see in the following. It follows from the definition of* u *that*

$$u'(t) = \left(\frac{2\alpha}{(t+1)^2} - \frac{2\alpha}{(t+1)^3} \right) \cos \sqrt{a}t - \left(\frac{\sqrt{a}\alpha}{(t+1)^2} + \frac{4\alpha}{\sqrt{a}(t+1)^3} \right) \sin \sqrt{a}t$$

and hence, we get $u(0) = u'(0) = 0$. *Moreover, we obtain*

$$\left| u'(t) \right| \leq \frac{2+\sqrt{a}}{(t+1)^2} \alpha + \left(\frac{4}{\sqrt{a}} - 2 \right) \frac{\alpha}{(t+1)^3}$$

and

$$\begin{aligned} \int_0^\infty \left| u'(\tau) \right| d\tau &= \int_0^\infty \frac{2+\sqrt{a}}{(\tau+1)^2} \alpha d\tau + \int_0^\infty \left(\frac{4}{\sqrt{a}} - 2 \right) \frac{\alpha}{(\tau+1)^3} d\tau \\ &= \left(2 + \sqrt{a} \right) \alpha + \left(\frac{2}{\sqrt{a}} - 1 \right) \alpha \\ &= L. \end{aligned}$$

For any given $\varepsilon > 0$, *if we choose the constant* α *such that* $0 < \alpha \leq \frac{\sqrt{a}\varepsilon}{a\sqrt{a}+4a+2\sqrt{a}+12}$, *then we can easily see that*

$$\begin{aligned} &\left| u''(t) + au(t) \right| \\ &\leq \Bigg| \left(-\frac{8}{(t+1)^3} + \frac{6}{(t+1)^4} \right) \alpha \cos \sqrt{a}t \\ &+ \left(\frac{4\sqrt{a}}{(t+1)^3} + \frac{1}{\sqrt{a}} \frac{12}{(t+1)^4} \right) \alpha \sin \sqrt{a}t - a\alpha \Bigg| \\ &\leq \left(\frac{8}{(t+1)^3} - \frac{6}{(t+1)^4} \right) \alpha + \left(\frac{4\sqrt{a}}{(t+1)^3} + \frac{1}{\sqrt{a}} \frac{12}{(t+1)^4} \right) \alpha + a\alpha \\ &= \frac{a\sqrt{a} + 4a + 2\sqrt{a} + 12}{\sqrt{a}} \alpha \\ &\leq \varepsilon \end{aligned}$$

for any $t \geq 0$.

Theorem 1.1.6. *[28] Given constants $L > 0$ and $t_0 \geq 0$, assume that $\psi : [t_0, \infty) \to (0, \infty)$ is a monotone increasing and differentiable function. If a function $u \in U(L; t_0)$ satisfies the inequality (1.1.6) for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 \in U(L; t_0)$ of the differential equation (1.1.2) such that*

$$|u(t) - u_0(t)| \leq \sqrt{\frac{2L\varepsilon}{\psi(t_0)}} \tag{1.1.9}$$

for any $t \geq t_0$.

Proof. We multiply (1.1.6) with $|u'(t)|$ to get

$$-\varepsilon|u'(t)| \leq u'(t)u''(t) + u(t)u'(t) + \psi(t)u(t)u'(t) \leq \varepsilon|u'(t)|$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii) that

$$-\varepsilon \int_{t_0}^{t} |u'(\tau)|d\tau \leq \frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \int_{t_0}^{t} \psi(\tau)u(\tau)u'(\tau)d\tau \leq \varepsilon \int_{t_0}^{t} |u'(\tau)|d\tau$$

for any $t \geq t_0$.
Integrating by parts, the last inequalities together with (iii) yield

$$-\varepsilon L \leq \frac{1}{2}u'(t)^2 + \frac{1}{2}u(t)^2 + \frac{1}{2}\psi(t)u(t)^2 - \frac{1}{2}\int_{t_0}^{t} \psi'(\tau)u(\tau)^2 d\tau \leq \varepsilon L$$

for all $t \geq t_0$. Then we have

$$\frac{1}{2}\psi(t)u(t)^2 \leq \frac{1}{2}\int_{t_0}^{t} \psi'(\tau)u(\tau)^2 d\tau + \varepsilon L \leq \varepsilon L + \int_{t_0}^{t} \frac{\psi'(\tau)}{\psi(\tau)}u(\tau)^2 \frac{\psi(\tau)}{2} d\tau$$

for any $t \geq t_0$.
Applying Lemma 1.1.1, we obtain

$$\frac{1}{2}\psi(t)u(t)^2 \leq \varepsilon L \exp\left(\int_{t_0}^{t} \frac{\psi'(\tau)}{\psi(\tau)} d\tau\right) = \varepsilon L \frac{\psi(t)}{\psi(t_0)}$$

for all $t \geq t_0$, since $\psi : [t_0, \infty) \to (0, \infty)$ is a monotone increasing function. Hence, it holds that

$$|u(t)| \leq \sqrt{\frac{2L\varepsilon}{\psi(t_0)}}$$

for any $t \geq t_0$. Obviously, $u_0(t) \equiv 0$ satisfies the equation (1.1.2), $u_0 \in U(L; t_0)$, as well as the inequality (1.1.9) for all $t \geq t_0$. □

Corollary 1.1.7. *[28] Given constants $L > 0$ and $t_0 \geq 0$, assume that $\phi : [t_0, \infty) \to (1, \infty)$ is a monotone increasing and differentiable function with $\phi(t_0) = 2$. If a function $u \in U(L; t_0)$ satisfies the inequality*

$$\left|u''(t) + \phi(t)u(t)\right| \leq \varepsilon$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 \in U(L; t_0)$ of the differential equation (1.1.5) such that

$$|u(t) - u_0(t)| \leq \sqrt{2L\varepsilon}$$

for any $t \geq t_0$.

If we set $\phi(t) := -\psi(t)$, then the following corollary is an immediate consequence of Theorem 1.1.6.

Corollary 1.1.8. *[28] Given constants $L > 0$ and $t_0 \geq 0$, assume that $\phi : [t_0, \infty) \to (-\infty, 0)$ is a monotone decreasing and differentiable function with $\phi(t_0) = -1$. If a function $u \in U(L; t_0)$ satisfies the inequality*

$$\left|u''(t) + \big(1 - \phi(t)\big)u(t)\right| \leq \varepsilon$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 \in U(L; t_0)$ of the differential equation

$$u''(t) + \big(1 - \phi(t)\big)u(t) = 0$$

such that

$$|u(t) - u_0(t)| \leq \sqrt{2L\varepsilon}$$

for any $t \geq t_0$.

Example 1.1.2. *[28] Let $\phi : [0, \infty) \to (-\infty, 0)$ be a monotone decreasing function defined by $\phi(t) := e^{-t} - 2$ for all $t \geq 0$. Then, we have $\phi(0) = -1$. Assume that a twice continuously differentiable function $u : [0, \infty) \to \mathbb{R}$ satisfies $u(0) = u'(0) = 0$, $\int_0^\infty |u'(\tau)| d\tau \leq L$, and*

$$\left|u''(t) + \big(1 - \phi(t)\big)u(t)\right| = \left|u''(t) + \big(3 - e^{-t}\big)u(t)\right| \leq \varepsilon$$

for all $t \geq 0$ and for some $\varepsilon > 0$ and $L > 0$. According to Corollary 1.1.8, there exists a solution $u_0 : [0, \infty) \to \mathbb{R}$ of the differential equation,
$y''(t) + \big(3 - e^{-t}\big)y(t) = 0$, such that

$$|u(t) - u_0(t)| \leq \sqrt{2L\varepsilon}$$

for any $t \geq 0$.
Indeed, if we define a function $u : [0, \infty) \to \mathbb{R}$ by

$$u(t) := \frac{\alpha}{(t+1)^3}\sin t + \frac{1}{2}\frac{\alpha}{(t+1)^2}\cos t - \frac{\alpha}{2},$$

where α is a real number with $|\alpha| \leq \frac{2}{43}\varepsilon$, then u satisfies the conditions stated in the first part of this example, as we see in the following. It follows from the definition of u that

$$u'(t) = -\frac{3\alpha}{(t+1)^4}\sin t - \frac{1}{2}\frac{\alpha}{(t+1)^2}\sin t$$

and hence, we get $u(0) = u'(0) = 0$. Moreover, we obtain

$$\big|u'(t)\big| \leq \frac{3|\alpha|}{(t+1)^4} + \frac{1}{2}\frac{|\alpha|}{(t+1)^2}$$

and

$$\int_0^\infty \big|u'(\tau)\big|d\tau \leq \int_0^\infty \frac{3|\alpha|}{(\tau+1)^4}d\tau + \int_0^\infty \frac{1}{2}\frac{|\alpha|}{(\tau+1)^2}d\tau =: L < \infty.$$

We can see that

$$\begin{aligned}
&\big|u''(t) + \big(3 - e^{-t}\big)u(t)\big| \\
&\leq \left|\frac{12\alpha}{(t+1)^5}\sin t - \frac{3\alpha}{(t+1)^4}\cos t + \big(4 - e^{-t}\big)\frac{\alpha}{(t+1)^3}\sin t \right. \\
&\quad \left. + \frac{2 - e^{-t}}{2}\frac{\alpha}{(t+1)^2}\cos t - \frac{3 - e^{-t}}{2}\alpha\right| \\
&\leq \frac{12|\alpha|}{(t+1)^5} + \frac{3|\alpha|}{(t+1)^4} + \frac{4|\alpha|}{(t+1)^3} + \frac{|\alpha|}{(t+1)^2} + \frac{3}{2}|\alpha| \\
&\leq \frac{43}{2}|\alpha| \\
&\leq \varepsilon
\end{aligned}$$

for any $t \geq 0$.

1.2 Hyers-Ulam Stability of Nonlinear Differential Equations of Second Order

In the following theorems, we investigate the Hyers-Ulam stability of the nonlinear differential equation

$$u''(t) + F(t, u(t)) = 0. \tag{1.2.1}$$

Theorem 1.2.1. *[28] Given constants $L > 0$ and $t_0 \geq 0$, assume that $F : [t_0, \infty) \times \mathbb{R} \to (0, \infty)$ is a function satisfying $F'(t, u(t))/F(t, u(t)) > 0$ and $F(t, 0) = 1$ for all $t \geq t_0$ and $u \in U(L; t_0)$. If a function $u : [t_0, \infty) \to [0, \infty)$ satisfies $u \in U(L; t_0)$ and the inequality*

$$\left| u''(t) + F(t, u(t)) \right| \leq \varepsilon \tag{1.2.2}$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 : [t_0, \infty) \to [0, \infty)$ of the differential equation (1.2.2) *such that*

$$|u(t) - u_0(t)| \leq L\varepsilon$$

for any $t \geq t_0$.

Proof. We multiply (1.2.2) with $|u'(t)|$ to get

$$-\varepsilon|u'(t)| \leq u'(t)u''(t) + F(t, u(t))u'(t) \leq \varepsilon|u'(t)|$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii) that

$$-\varepsilon \int_{t_0}^{t} |u'(\tau)|d\tau \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} F(\tau, u(\tau))u'(\tau)d\tau \leq \varepsilon \int_{t_0}^{t} |u'(\tau)|d\tau$$

for any $t \geq t_0$.
Integrating by parts and using (iii), the last inequalities yield

$$-\varepsilon L \leq \frac{1}{2}u'(t)^2 + F(t, u(t))u(t) - \int_{t_0}^{t} F'(\tau, u(\tau))u(\tau)d\tau \leq \varepsilon L$$

for all $t \geq t_0$. Then we have

$$\begin{aligned} F(t, u(t))u(t) &\leq \varepsilon L + \int_{t_0}^{t} F'(\tau, u(\tau))u(\tau)d\tau \\ &\leq \varepsilon L + \int_{t_0}^{t} \frac{F'(\tau, u(\tau))}{F(\tau, u(\tau))} F(\tau, u(\tau))u(\tau)d\tau \end{aligned}$$

for any $t \geq t_0$.
Applying Lemma 1.1.1, we obtain

$$F(t,u(t))u(t) \leq \varepsilon L \exp\left(\int_{t_0}^{t} \frac{F'(\tau,u(\tau))}{F(\tau,u(\tau))} d\tau\right) = \varepsilon L F(t,u(t))$$

for all $t \geq t_0$. Hence, it holds that $|u(t)| \leq L\varepsilon$ for any $t \geq t_0$. Obviously, $u_0(t) \equiv 0$ satisfies the equation (1.2.1) and $u_0 \in U(L;t_0)$ such that

$$|u(t) - u_0(t)| \leq L\varepsilon$$

for all $t \geq t_0$. □

In the following theorem, we investigate the Hyers-Ulam stability of the Emden-Fowler nonlinear differential equation of second order

$$u''(t) + h(t)u(t)^{\alpha} = 0 \tag{1.2.3}$$

for the case where α is a positive odd integer.

Theorem 1.2.2. *[28] Given constants $L > 0$ and $t_0 \geq 0$, assume that $h : [t_0,\infty) \to (0,\infty)$ is a differentiable function. Let α be an odd integer larger than 0. If a function $u : [t_0,\infty) \to [0,\infty)$ satisfies $u \in U(L;t_0)$ and the inequality*

$$\left|u''(t) + h(t)u(t)^{\alpha}\right| \leq \varepsilon \tag{1.2.4}$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 : [t_0,\infty) \to [0,\infty)$ of the differential equation (1.2.3) such that

$$|u(t) - u_0(t)| \leq \left(\frac{\beta L\varepsilon}{h(t_0)}\right)^{1/\beta}$$

for any $t \geq t_0$, where $\beta := \alpha + 1$.

Proof. We multiply (1.2.4) with $|u'(t)|$ to get

$$-\varepsilon|u'(t)| \leq u'(t)u''(t) + h(t)u(t)^{\alpha}u'(t) \leq \varepsilon|u'(t)|$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii) that

$$-\varepsilon\int_{t_0}^{t}|u'(\tau)|d\tau \leq \frac{1}{2}u'(t)^2 + \int_{t_0}^{t} h(\tau)u(\tau)^{\alpha}u'(\tau)d\tau \leq \varepsilon\int_{t_0}^{t}|u'(\tau)|d\tau$$

for any $t \geq t_0$.
Integrating by parts and using (iii), the last inequalities yield

$$-\varepsilon L \leq \frac{1}{2}u'(t)^2 + h(t)\frac{u(t)^{\alpha+1}}{\alpha+1} - \int_{t_0}^{t} h'(\tau)\frac{u(\tau)^{\alpha+1}}{\alpha+1}d\tau \leq \varepsilon L$$

for all $t \geq t_0$. for all $t \geq t_0$. Then we have

$$\begin{aligned} h(t)\frac{u(t)^{\alpha+1}}{\alpha+1} &\leq \varepsilon L + \int_{t_0}^{t} h'(\tau)\frac{u(\tau)^{\alpha+1}}{\alpha+1}d\tau \\ &\leq \varepsilon L + \int_{t_0}^{t} \frac{h'(\tau)}{h(\tau)}h(\tau)\frac{u(\tau)^{\alpha+1}}{\alpha+1}d\tau \end{aligned}$$

for any $t \geq t_0$.
Applying Lemma 1.1.1, we obtain

$$h(t)\frac{u(t)^{\alpha+1}}{\alpha+1} \leq \varepsilon L \exp\left(\int_{t_0}^{t} \frac{h'(\tau)}{h(\tau)}d\tau\right) \leq \varepsilon L\frac{h(t)}{h(t_0)}$$

for all $t \geq t_0$, from which we have

$$u(t)^{\alpha+1} \leq \frac{(\alpha+1)L\varepsilon}{h(t_0)}$$

for all $t \geq t_0$. Hence, it holds that

$$|u(t)| \leq \left(\frac{\beta L\varepsilon}{h(t_0)}\right)^{1/\beta}$$

for any $t \geq t_0$, where we set $\beta = \alpha + 1$. Obviously, $u_0(t) \equiv 0$ satisfies the equation (1.2.3) and $u_0 \in U(L; t_0)$. Moreover, we get

$$|u(t) - u_0(t)| \leq \left(\frac{\beta L\varepsilon}{h(t_0)}\right)^{1/\beta}$$

for all $t \geq t_0$. □

Given constants $L \geq 0$, $M > 0$, and $t_0 \geq 0$, let $U(L; M; t_0)$ denote the set of all functions $u : [t_0, \infty) \to \mathbb{R}$ with the following properties:

(i') u is twice continuously differentiable;

(ii') $u(t_0) = u'(t_0) = 0$;

(iii') $|u(t)| \leq L$ for all $t \geq t_0$;

(iv') $\int_{t_0}^{\infty} |u'(\tau)| d\tau \leq M$ for all $t \geq t_0$.

We now investigate the Hyers-Ulam stability of the differential equation of the form

$$u''(t) + u(t) + h(t)u(t)^{\beta} = 0, \tag{1.2.5}$$

where β is a positive odd integer.

Theorem 1.2.3. *[28] Given constants $L \geq 0$, $M > 0$ and $t_0 \geq 0$, assume that $h : [t_0, \infty) \to [0, \infty)$ is a function satisfying $C := \int_{t_0}^{\infty} |h'(\tau)| d\tau < \infty$. Let β be an odd integer larger than 0. If a function $u \in U(L; M; t_0)$ satisfies the inequality*

$$\left| u''(t) + u(t) + h(t)u(t)^{\beta} \right| \leq \varepsilon \tag{1.2.6}$$

for all $t \geq t_0$ and for some $\varepsilon > 0$, then there exists a solution $u_0 : [t_0, \infty) \to \mathbb{R}$ of the differential equation (1.2.5) *such that*

$$|u(t) - u_0(t)| \leq \sqrt{2M\varepsilon} \exp\left(\frac{CL^{\beta-1}}{\beta+1} \right)$$

for any $t \geq t_0$.

Proof. We multiply (1.2.6) with $|u'(t)|$ to get

$$-\varepsilon |u'(t)| \leq u'(t)u''(t) + u(t)u'(t) + h(t)u(t)^{\beta}u'(t) \leq \varepsilon |u'(t)|$$

for all $t \geq t_0$. If we integrate each term of the last inequalities from t_0 to t, then it follows from (ii') that

$$-\varepsilon \int_{t_0}^{t} |u'(\tau)| d\tau \leq \frac{1}{2} u'(t)^2 + \frac{1}{2} u(t)^2 + \int_{t_0}^{t} h(\tau)u(\tau)^{\beta}u'(\tau) d\tau \leq \varepsilon \int_{t_0}^{t} |u'(\tau)| d\tau$$

for any $t \geq t_0$.
Integrating by parts and using (ii') and (iv'), the last inequalities yield

$$-\varepsilon M \leq \frac{1}{2} u'(t)^2 + \frac{1}{2} u(t)^2 + h(t) \frac{1}{\beta+1} u(t)^{\beta+1} - \frac{1}{\beta+1} \int_{t_0}^{t} h'(\tau)u(\tau)^{\beta+1} d\tau \leq \varepsilon M$$

for all $t \geq t_0$. Then it follows from (iii') that

$$\begin{aligned}
\frac{1}{2}u(t)^2 &\leq \varepsilon M + \frac{1}{\beta+1}\int_{t_0}^{t} h'(\tau)u(\tau)^{\beta+1}d\tau \\
&\leq \varepsilon M + \frac{2}{\beta+1}\int_{t_0}^{t} \frac{1}{2}u(\tau)^2 h'(\tau)u(\tau)^{\beta-1}d\tau \\
&\leq \varepsilon M + \frac{2}{\beta+1}\int_{t_0}^{t} \frac{1}{2}u(\tau)^2 |h'(\tau)||u(\tau)|^{\beta-1}d\tau \\
&\leq \varepsilon M + \frac{2L^{\beta-1}}{\beta+1}\int_{t_0}^{t} \frac{1}{2}u(\tau)^2 \big|h'(\tau)\big|d\tau
\end{aligned}$$

for any $t \geq t_0$.
Applying Lemma 1.1.1, we obtain

$$\frac{1}{2}u(t)^2 \leq \varepsilon M \exp\left(\int_{t_0}^{t} \frac{2L^{\beta-1}}{\beta+1}|h'(\tau)|d\tau\right) \leq \varepsilon M \exp\left(\frac{2CL^{\beta-1}}{\beta+1}\right)$$

for all $t \geq t_0$. Hence, it holds that

$$|u(t)| \leq \sqrt{2M\varepsilon}\exp\left(\frac{CL^{\beta-1}}{\beta+1}\right)$$

for any $t \geq t_0$. Obviously, $u_0(t) \equiv 0$ satisfies the equation (1.2.5) and $u_0 \in U(L; M; t_0)$. Furthermore, we get

$$|u(t) - u_0(t)| \leq \sqrt{2M\varepsilon}\exp\left(\frac{CL^{\beta-1}}{\beta+1}\right)$$

for all $t \geq t_0$. □

1.3 Hyers-Ulam Stability of Differential Equations with Boundary Conditions

Lemma 1.3.1. *[29] Let $I = [a, b]$ be a closed interval with $-\infty < a < b < \infty$. If $y \in C^2(I, \mathbb{R})$ and $y(a) = 0 = y(b)$, then*

$$\max_{x\in I}|y(x)| \leq \frac{(b-a)^2}{8}\max_{x\in I}|y''(x)|.$$

Proof. Let $M := \max_{x\in I} |y(x)|$. Since $y(a) = 0 = y(b)$, there exists $x_0 \in (a, b)$ such that $|y(x_0)| = M$. By the Taylor's theorem, we have

$$y(a) = y(x_0) + y'(x_0)(a - x_0) + \frac{y''(\xi)}{2}(a - x_0)^2,$$
$$y(b) = y(x_0) + y'(x_0)(b - x_0) + \frac{y''(\eta)}{2}(b - x_0)^2$$

for some $\xi, \eta \in [a, b]$. Since $y(a) = y(b) = 0$ and $y'(x_0) = 0$, we get

$$|y''(\xi)| = \frac{2M}{(a - x_0)^2}, \quad |y''(\eta)| = \frac{2M}{(b - x_0)^2}.$$

If $x_0 \in \big(a, (a + b)/2\big]$, then we have

$$\frac{2M}{(a - x_0)^2} \geq \frac{2M}{\left(\frac{b-a}{2}\right)^2} = \frac{8M}{(b - a)^2}.$$

If $x_0 \in \big[(a + b)/2, b\big)$, then we have

$$\frac{2M}{(b - x_0)^2} \geq \frac{2M}{\left(\frac{b-a}{2}\right)^2} = \frac{8M}{(b - a)^2}.$$

Hence, we obtain

$$\max_{x\in I} |y''(x)| \geq \frac{8M}{(b - a)^2} = \frac{8}{(b - a)^2} \max_{x\in I} |y(x)|.$$

Therefore,

$$\max_{x\in I} |y(x)| \leq \frac{(b - a)^2}{8} \max_{x\in I} |y''(x)|,$$

which ends the proof. □

Lemma 1.3.2. *[29] Let $I = [a, b]$ be a closed interval with $-\infty < a < b < \infty$. If $y \in C^2(I, \mathbb{R})$ and $y(a) = 0 = y'(a)$, then*

$$\max_{x\in I} |y(x)| \leq \frac{(b - a)^2}{2} \max_{x\in I} |y''(x)|.$$

Proof. By the Taylor's theorem, we have

$$y(x) = y(a) + y'(a)(x - a) + \frac{y''(\xi)}{2}(x - a)^2$$

for some $\xi \in [a, b]$. Since $y(a) = y'(a) = 0$ and $(x - a)^2 \leq (b - a)^2$, we get

$$|y(x)| \leq \frac{|y''(\xi)|}{2}(b - a)^2$$

for any $x \in I$. Thus, we obtain

$$\max_{x \in I} |y(x)| \leq \frac{(b - a)^2}{2} \max_{x \in I} |y''(x)|,$$

which completes the proof. □

In the following theorems, we prove the Hyers-Ulam stability of the following linear differential equation

$$y''(x) + \beta(x)y(x) = 0 \tag{1.3.1}$$

with boundary conditions

$$y(a) = 0 = y(b) \tag{1.3.2}$$

or with initial conditions

$$y(a) = 0 = y'(a) \tag{1.3.3}$$

where $I = [a, b]$, $y \in C^2(I, \mathbb{R})$, $\beta \in C(I, \mathbb{R})$, and $-\infty < a < b < \infty$.

Theorem 1.3.3. *[29] Given a closed interval $I = [a, b]$, let $\beta \in C(I, \mathbb{R})$ be a function satisfying $\max_{x \in I} |\beta(x)| < 8/(b - a)^2$. If a function $y \in C^2(I, \mathbb{R})$ satisfies the inequality*

$$|y''(x) + \beta(x)y(x)| \leq \varepsilon, \tag{1.3.4}$$

for all $x \in I$ and for some $\varepsilon \geq 0$, as well as the boundary conditions in (1.3.2)*, then there exist a constant $K > 0$ and a solution $y_0 \in C^2(I, \mathbb{R})$ of the differential equation* (1.3.1) *with the boundary conditions in* (1.3.2) *such that*

$$|y(x) - y_0(x)| \leq K\varepsilon$$

for any $x \in I$.

Proof. By Lemma 1.3.1, we have

$$\max_{x\in I}|y(x)| \le \frac{(b-a)^2}{8}\max_{x\in I}|y''(x)|.$$

Thus, it follows from (1.3.4) that

$$\begin{aligned}\max_{x\in I}|y(x)| &\le \frac{(b-a)^2}{8}\max_{x\in I}|y''(x)+\beta(x)y(x)| + \frac{(b-a)^2}{8}\max_{x\in I}|\beta(x)|\max_{x\in I}|y(x)| \\ &\le \frac{(b-a)^2}{8}\varepsilon + \frac{(b-a)^2}{8}\max_{x\in I}|\beta(x)|\max_{x\in I}|y(x)|.\end{aligned}$$

Let $C := \frac{(b-a)^2}{8}$ and $K := \frac{C}{1-C\max|\beta(x)|}$. Obviously, $y_0 \equiv 0$ is a solution of (1.3.1) with the boundary conditions in (1.3.2) and

$$|y(x) - y_0(x)| \le K\varepsilon$$

for any $x \in I$. □

Theorem 1.3.4. *[29] Given a closed interval $I = [a,b]$, let $\beta : I \to \mathbb{R}$ be a function satisfying $\max_{x\in I}|\beta(x)| < 2/(b-a)^2$. If a function $y \in C^2(I,\mathbb{R})$ satisfies the inequality* (1.3.4) *for all $x \in I$ and for some $\varepsilon \ge 0$ as well as the initial conditions in* (1.3.3)*, then there exist a solution $y_0 \in C^2(I,\mathbb{R})$ of the differential equation* (1.3.1) *with the initial conditions in* (1.3.3) *and a constant $K > 0$ such that*

$$|y(x) - y_0(x)| \le K\varepsilon$$

for any $x \in I$.

Proof. On account of Lemma 1.3.2, we have

$$\max_{x\in I}|y(x)| \le \frac{(b-a)^2}{2}\max_{x\in I}|y''(x)|.$$

Thus, it follows from (1.3.4) that

$$\begin{aligned}\max_{x\in I}|y(x)| &\le \frac{(b-a)^2}{2}\max_{x\in I}|y''(x)+\beta(x)y(x)| + \frac{(b-a)^2}{2}\max_{x\in I}|\beta(x)|\max_{x\in I}|y(x)| \\ &\le \frac{(b-a)^2}{2}\varepsilon + \frac{(b-a)^2}{2}\max_{x\in I}|\beta(x)|\max_{x\in I}|y(x)|.\end{aligned}$$

Let $C := \frac{(b-a)^2}{2}$ and $K := \frac{C}{1-C\max|\beta(x)|}$. Obviously, $y_0 \equiv 0$ is a solution of (1.3.1) with the initial conditions in (1.3.3) and

$$|y(x) - y_0(x)| \leq K\varepsilon$$

for all $x \in I$. □

In the following theorems, we investigate the Hyers-Ulam stability of the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{1.3.5}$$

with boundary conditions

$$y(a) = 0 = y(b) \tag{1.3.6}$$

or with initial conditions

$$y(a) = 0 = y'(a) \tag{1.3.7}$$

where $y \in C^2(I, \mathbb{R})$, $p \in C^1(I, \mathbb{R})$, $q \in C(I, \mathbb{R})$, and $I = [a, b]$ with $-\infty < a < b < \infty$.
Let us define a function $\beta : I \to \mathbb{R}$ by

$$\beta(x) := q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2$$

for all $x \in I$.

Theorem 1.3.5. *[29] Assume that there exists a constant $L \geq 0$ with*

$$-L \leq \int_a^x p(\tau)d\tau \leq L \tag{1.3.8}$$

for any $x \in I$ and $\max_{x\in I} |\beta(x)| < 8/(b-a)^2$. If a function $y \in C^2(I, \mathbb{R})$ satisfies the inequality

$$|y''(x) + p(x)y'(x) + q(x)y(x)| \leq \varepsilon \tag{1.3.9}$$

for all $x \in I$ and for some $\varepsilon \geq 0$ as well as the boundary conditions in (1.3.6), then there exist a constant $K > 0$ and a solution $y_0 \in C^2(I, \mathbb{R})$ of the differential equation (1.3.5) with the boundary conditions in (1.3.6) such that

$$|y(x) - y_0(x)| \leq Ke^L\varepsilon$$

for any $x \in I$.

Proof. Suppose $y \in C^2(I, \mathbb{R})$ satisfies the inequality (1.3.9) for all $x \in I$. Let us define

$$u(x) := y''(x) + p(x)y'(x) + q(x)y(x), \tag{1.3.10}$$

$$z(x) := y(x) \exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right) \tag{1.3.11}$$

for all $x \in I$. By (1.3.10) and (1.3.11), we obtain

$$z''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z(x) = u(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)$$

for all $x \in I$.
Now, it follows from (1.3.8) and (1.3.9) that

$$\left|z''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z(x)\right| = \left|u(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right| \leq \varepsilon e^{L/2},$$

that is,

$$|z''(x) + \beta(x)z(x)| \leq \varepsilon e^{L/2}$$

for any $x \in I$. Moreover, it follows from (1.3.11) that

$$z(a) = 0 = z(b).$$

In view of Theorem 1.3.3, there exists a constant $K > 0$ and a function $z_0 \in C^2(I, \mathbb{R})$ such that

$$z_0''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z_0(x) = 0, \tag{1.3.12}$$
$$z_0(a) = 0 = z_0(b)$$

and

$$|z(x) - z_0(x)| \leq K\varepsilon e^{L/2} \tag{1.3.13}$$

for all $x \in I$.
We now set

$$y_0(x) := z_0(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right). \tag{1.3.14}$$

Then, since

$$y_0'(x) = z_0'(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) - \frac{1}{2}p(x)z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big), \tag{1.3.15}$$

$$\begin{aligned} y_0''(x) = {} & z_0''(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) - p(x)z_0'(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ & - \frac{1}{2}p'(x)z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ & + \frac{1}{4}p(x)^2 z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big), \end{aligned} \tag{1.3.16}$$

it follows from (1.3.12), (1.3.14), (1.3.15), and (1.3.16) that

$$\begin{aligned} & y_0''(x) + p(x)y_0'(x) + q(x)y_0(x) \\ & = \Big(z_0''(x) + \Big(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\Big)z_0(x)\Big)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ & = 0 \end{aligned}$$

for all $x \in I$. Hence, y_0 satisfies (1.3.5) and the boundary conditions in (1.3.6). Finally, it follows from (1.3.8) and (1.3.13) that

$$\begin{aligned} |y(x) - y_0(x)| & = \Big|z(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) - z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big)\Big| \\ & = |z(x) - z_0(x)|\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ & \le K\varepsilon e^{L/2}\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\ & \le Ke^L\varepsilon \end{aligned}$$

for all $x \in I$. □

Theorem 1.3.6. *[29] Assume that there exists a constant $L \ge 0$ such that* (1.3.8) *holds for all $x \in I$. Assume moreover that $\max_{x\in I}|\beta(x)| < 2/(b-a)^2$. If a function $y \in C^2(I,\mathbb{R})$ satisfies the inequality* (1.3.9) *for all $x \in I$ and for some $\varepsilon \ge 0$ as well as the initial conditions in* (1.3.7)*, then there exist a constant $K > 0$ and a solution $y_0 \in C^2(I,\mathbb{R})$ of the differential equation* (1.3.5) *with the initial conditions in* (1.3.7) *such that*

$$|y(x) - y_0(x)| \le Ke^L\varepsilon$$

for any $x \in I$.

Proof. Suppose $y \in C^2(I, \mathbb{R})$ satisfies the inequality (1.3.9) for any $x \in I$. Let us define $u(x)$ and $z(x)$ as in (1.3.10) and (1.3.11), respectively. By (1.3.10) and (1.3.11), we obtain

$$z''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z(x) = u(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)$$

for all $x \in I$.
Now, it follows from (1.3.8) and (1.3.9) that

$$\left|z''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z(x)\right| = \left|u(x)\exp\left(\frac{1}{2}\int_a^x p(\tau)d\tau\right)\right| \leq \varepsilon e^{L/2},$$

that is,

$$|z''(x) + \beta(x)z(x)| \leq \varepsilon e^{L/2}$$

for all $x \in I$. Furthermore, in view of (1.3.11), we have

$$z(a) = 0 = z'(a).$$

By Theorem 1.3.4, there exists a constant $K > 0$ and a function $z_0 \in C^2(I, \mathbb{R})$ such that

$$z_0''(x) + \left(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\right)z_0(x) = 0,$$
$$z_0(a) = 0 = z_0'(a)$$

and

$$|z(x) - z_0(x)| \leq K\varepsilon e^{L/2}$$

for any $x \in I$.
We now set

$$y_0(x) := z_0(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right).$$

Moreover, since

$$y_0'(x) = z_0'(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right) - \frac{1}{2}p(x)z_0(x)\exp\left(-\frac{1}{2}\int_a^x p(\tau)d\tau\right)$$

and

$$
\begin{aligned}
y_0''(x) = & z_0''(x) \exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) - p(x)z_0'(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\
& - \frac{1}{2}p'(x)z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\
& + \frac{1}{4}p(x)^2 z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big),
\end{aligned}
$$

we have

$$
\begin{aligned}
& y_0''(x) + p(x)y_0'(x) + q(x)y_0(x) \\
& = \Big(z_0''(x) + \Big(q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p(x)^2\Big)z_0(x)\Big)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\
& = 0
\end{aligned}
$$

for any $x \in I$. Hence, y_0 satisfies (1.3.5) along with the initial conditions in (1.3.7). Finally, it follows that

$$
\begin{aligned}
|y(x) - y_0(x)| & = \Big|z(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) - z_0(x)\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big)\Big| \\
& = |z(x) - z_0(x)|\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\
& \leq K\varepsilon e^{L/2}\exp\Big(-\frac{1}{2}\int_a^x p(\tau)d\tau\Big) \\
& \leq Ke^L\varepsilon
\end{aligned}
$$

for all $x \in I$. □

In a similar way, we investigate the Hyers-Ulam stability of the differential equation

$$y''(x) + \frac{k'(x)}{k(x)}y'(x) + \frac{l(x)}{k(x)}y(x) = 0 \tag{1.3.17}$$

with boundary conditions

$$y(a) = 0 = y(b) \tag{1.3.18}$$

or with initial conditions

$$y(a) = 0 = y'(a) \tag{1.3.19}$$

where $y \in C^2(I,\mathbb{R})$, $k \in C^1(I,\mathbb{R}\backslash\{0\})$, $l \in C(I,\mathbb{R})$, and $-\infty < a < b < \infty$. Given a closed interval $I = [a,b]$, we set

$$\beta(x) := \frac{l(x)}{k(x)} - \frac{1}{2}\frac{d}{dx}\frac{k'(x)}{k(x)} - \frac{1}{4}\left(\frac{k'(x)}{k(x)}\right)^2$$

for all $x \in I$.

Theorem 1.3.7. *[29] Assume that there exists a constant $L \geq 0$ with*

$$-L \leq \int_a^x \frac{k'(\tau)}{k(\tau)} d\tau \leq L \tag{1.3.20}$$

for any $x \in I$ and $\max_{x\in I} |\beta(x)| < 8/(b-a)^2$. If a function $y \in C^2(I,\mathbb{R})$ satisfies the inequality

$$\left| y''(x) + \frac{k'(x)}{k(x)} y'(x) + \frac{l(x)}{k(x)} y(x) \right| \leq \varepsilon, \tag{1.3.21}$$

for all $x \in I$ and some $\varepsilon \geq 0$, as well as the boundary conditions in (1.3.18)*, then there exist a constant $K > 0$ and a solution $y_0 \in C^2(I,\mathbb{R})$ of the differential equation* (1.3.17) *with the boundary conditions in* (1.3.18) *such that*

$$|y(x) - y_0(x)| \leq K e^L \varepsilon$$

for any $x \in I$.

Proof. Suppose $y \in C^2(I,\mathbb{R})$ satisfies (1.3.21) for all $x \in I$. Let us define

$$u(x) := y''(x) + \frac{k'(x)}{k(x)} y'(x) + \frac{l(x)}{k(x)} y(x), \tag{1.3.22}$$

$$z(x) := y(x) \exp\left(\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)} d\tau\right) \tag{1.3.23}$$

for all $x \in I$. By (1.3.22) and (1.3.23), we obtain

$$z''(x) + \left(\frac{l(x)}{k(x)} - \frac{1}{2}\frac{d}{dx}\frac{k'(x)}{k(x)} - \frac{1}{4}\left(\frac{k'(x)}{k(x)}\right)^2\right) z(x) = u(x) \exp\left(\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)} d\tau\right).$$

Further, it follows from (1.3.20) and (1.3.21) that

$$\begin{aligned}\left| z''(x) + \left(\frac{l(x)}{k(x)} - \frac{1}{2}\frac{d}{dx}\frac{k'(x)}{k(x)} - \frac{1}{4}\left(\frac{k'(x)}{k(x)}\right)^2\right) z(x) \right| &= \left| u(x) \exp\left(\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)} d\tau\right) \right| \\ &\leq \varepsilon \exp\left(\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)} d\tau\right) \\ &\leq \varepsilon e^{L/2},\end{aligned}$$

that is,

$$|z''(x) + \beta(x)z(x)| \le \varepsilon e^{L/2}$$

for all $x \in I$. Moreover, it follows from (1.3.18) and (1.3.23) that

$$z(a) = 0 = z(b).$$

By Theorem 1.3.3, there exists a constant $K > 0$ and a function $z_0 \in C^2(I, \mathbb{R})$ such that

$$z_0''(x) + \left(\frac{l(x)}{k(x)} - \frac{1}{2}\frac{d}{dx}\frac{k'(x)}{k(x)} - \frac{1}{4}\left(\frac{k'(x)}{k(x)}\right)^2\right)z_0(x) = 0,$$
$$z_0(a) = 0 = z_0(b)$$

and

$$|z(x) - z_0(x)| \le K\varepsilon e^{L/2}$$

for any $x \in I$.
We now set

$$y_0(x) := z_0(x)\exp\left(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\right).$$

Then, since

$$y_0'(x) = z_0'(x)\exp\left(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\right) - \frac{1}{2}\frac{k'(x)}{k(x)}z_0(x)\exp\left(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\right)$$

and

$$\begin{aligned} y_0''(x) = {} & z_0''(x)\exp\left(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\right) - \frac{k'(x)}{k(x)}z_0'(x)\exp\left(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\right) \\ & - \frac{1}{2}\left(\frac{k'(x)}{k(x)}\right)'z_0(x)\exp\left(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\right) \\ & + \frac{1}{4}\left(\frac{k'(x)}{k(x)}\right)^2 z_0(x)\exp\left(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\right), \end{aligned}$$

we have

$$\begin{aligned} & y_0''(x) + \frac{k'(x)}{k(x)}y_0'(x) + \frac{l(x)}{k(x)}y_0(x) \\ & = \left(z_0''(x) + \left(\frac{l(x)}{k(x)} - \frac{1}{2}\left(\frac{k'(x)}{k(x)}\right)' - \frac{1}{4}\left(\frac{k'(x)}{k(x)}\right)^2\right)z_0(x)\right)\exp\left(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\right) \\ & = 0. \end{aligned}$$

Hence, y_0 satisfies (1.3.17) along with the boundary conditions in (1.3.18). Finally, it follows that

$$\begin{aligned}|y(x)-y_0(x)| &= \left|z(x)\exp\Big(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\Big) - z_0(x)\exp\Big(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\Big)\right| \\ &= |z(x)-z_0(x)|\exp\Big(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\Big) \\ &\le K\varepsilon e^{L/2}\exp\Big(-\frac{1}{2}\int_a^x \frac{k'(\tau)}{k(\tau)}d\tau\Big) \\ &\le Ke^{L}\varepsilon\end{aligned}$$

for all $x \in I$. □

By a similar method as we applied to the proof of Theorem 1.3.6, we can prove the following theorem. Hence, we omit the proof.

Theorem 1.3.8. *[29] Assume that* $\max_{x\in I}|\beta(x)| < 2/(b-a)^2$ *and there exists a constant* $L \ge 0$ *for which the inequality* (1.3.20) *holds for all* $x \in I$*. If a function* $y \in C^2(I,\mathbb{R})$ *satisfies the inequality* (1.3.21) *for all* $x \in I$ *and for some* $\varepsilon \ge 0$ *as well as the boundary conditions in* (1.3.19)*, then there exist a constant* $K > 0$ *and a solution* $y_0 \in C^2(I,\mathbb{R})$ *of the differential equation* (1.3.17) *with the boundary conditions in* (1.3.19) *such that*

$$|y(x)-y_0(x)| \le Ke^{L}\varepsilon$$

for any $x \in I$*.*

Now, we give the definition of superstability with initial and boundary conditions.

Definition 1.3.9. *[18] Assume that for any function* $y \in C^n[a,b]$*, if* y *satisfies the differential inequality*

$$\big|\varphi\big(f,y,y',\ldots,y^{(n)}\big)\big| \le \epsilon$$

for all $x \in [a,b]$ *and for some* $\epsilon \ge 0$ *with initial(or boundary) conditions, then either* y *is a solution of the differential equation*

$$\varphi\big(f,y,y',\ldots,y^{(n)}\big) = 0 \tag{1.3.24}$$

or $|y(x)| \leq K\epsilon$ *for any* $x \in [a,b]$*, where* K *is a constant not depending on* y *explicitly. Then, we say that Eq.(1.3.24) has superstability with initial(or boundary) conditions.*

In the following theorem, we investigate the stability of differential equation of higher order in the form of

$$y^{(n)}(x) + \beta(x)y(x) = 0 \tag{1.3.25}$$

with initial conditions

$$y(a) = y^{'}(a) = \cdots = y^{(n-1)}(a) = 0, \tag{1.3.26}$$

where $n \in \mathbb{N}^{+}$, $y \in C^{n}[a,b], \beta \in C^{0}[a,b],\ -\infty < a < b < +\infty$.

Theorem 1.3.10. *[18] If* $\max|\beta(x)| < \frac{n!}{(b-a)^n}$*. Then (1.3.25) has the superstability with initial conditions (1.3.26).*

Proof. For every $\epsilon > 0$, $y \in C^{2}[a,b]$, if $\left|y^{(n)}(x) + \beta(x)y(x)\right| \leq \epsilon$ and $y(a) = y^{'}(a) = \cdots = y^{(n-1)}(a) = 0$. Similarly to the proof of Lemma 1.3.2,

$$y(x) = y(a) + y'(a)(x-a) + \cdots + \frac{y^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{y^{(n)}(\xi)}{n!}(x-a)^{n}.$$

Thus

$$|y(x)| = \left|\frac{y^{(n)}(\xi)}{n!}(x-a)^{n}\right| \leq \max\left|y^{(n)}(x)\right|\frac{(b-a)^n}{n!}$$

for every $x \in [a,b]$; so, we obtain

$$\begin{aligned}\max|y(x)| &\leq \frac{(b-a)^n}{n!}[\max|y^{(n)}(x) + \beta(x)y(x)|] + \frac{(b-a)^n}{n!}\max|\beta(x)y(x)| \\ &\leq \frac{(b-a)^n}{n!}\epsilon + \frac{(b-a)^n}{n!}\max|\beta(x)|\max|y(x)|.\end{aligned}$$

Let $\eta = \frac{(b-a)^n}{n!}\max|\beta(x)|$, $K = \frac{(b-a)^n}{n!(1-\eta)}$. It is easy to see that

$$|y(x)| \leq K\epsilon.$$

Hence (1.3.25) has superstability with initial condtions (1.3.26). □

In the following theorems, we investigate the superstability of the differential equation

$$y^{(n)}(x) + a_{n-1}y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = 0 \tag{1.3.27}$$

with initial conditions

$$y(a) = y^{'}(a) = \cdots = y^{(n-1)}(a) = 0, \tag{1.3.28}$$

where $y \in C^{n}(I, \mathbb{C}),\ a_i \in \mathbb{R}(i = 0, 1, \cdots, n-1),\ I = [a,b],\ -\infty < a < b < +\infty$.

Lemma 1.3.11. *[19] Assume that* $y \in C^1(I, \mathbb{C})$ *and* $C \in \{z \in \mathbb{C} | \, |z| < \frac{1}{b-a}\}$. *If*

$$|y'(x) - Cy(x)| \leq \varepsilon$$

with $y(a) = 0$, *then there exists a constant* $K > 0$ *such that*

$$|y(x)| \leq K\varepsilon.$$

Proof. Let $y(x) = A(x) + i \cdot B(x)$, where i denotes imaginary unit and $A(x), B(x) \in C^1(I, \mathbb{R})$. Since $y(a) = 0$, we have

$$A(a) = 0 \text{ and } B(a) = 0;$$

By Taylor formula, we obtain

$$\begin{aligned} \max |A(x)| &\leq (b-a) \max |A'(x) - CA(x)| + |C| \cdot (b-a) \max |A(x)| \\ &\leq (b-a) \max |y'(x) - Cy(x)| + |C| \cdot (b-a) \max |A(x)| \\ &\leq (b-a)\varepsilon + |C| \cdot (b-a) \max |A(x)| \end{aligned}$$

and

$$\max |B(x)| \leq (b-a)\varepsilon + |C| \cdot (b-a) \max |B(x)| .$$

Since $C \in \{z \in \mathbb{C} | \, |z| < \frac{1}{b-a}\}$, there exists a constant K such that

$$\max |y(x)| \leq \sqrt{\max |A(x)|^2 + \max |B(x)|^2} \leq K\varepsilon.$$

□

Theorem 1.3.12. *[19] If all the roots of the characteristic equation are in the disc* $\{z \in \mathbb{C} | |z| < \frac{1}{b-a}\}$, *then (1.3.27) has superstability with initial conditions (1.3.28).*

Proof. Assume that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the roots of the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

Define $g_1(x) = y'(x) - \lambda_1 y(x)$ and $g_i(x) = g'_{i-1}(x) - \lambda_i g_{i-1}(x) (i = 2, 3, \cdots, n-1)$, thus

$$\left| g'_{n-1}(x) - \lambda_n g_{n-1}(x) \right| = \left| y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) \right| \leq \varepsilon,$$

and $g_i(a) = 0$ for every $i = 1, 2, \cdots, n-1$.
Since the absolute value of $\lambda_n < \frac{1}{b-a}$ and $g_{n-1}(a) = 0$, it follows from Lemma 1.3.11 that there exists a $K_1 > 0$ such that

$$|g_{n-1}(x)| \leq K_1\varepsilon.$$

Recall $g_{n-1}(x) = g'_{n-2}(x) - \lambda_{n-1} g_{n-2}(x)$, we have

$$\left| g'_{n-2}(x) - \lambda_{n-1} g_{n-2}(x) \right| \leq K_1 \varepsilon.$$

By an argument similar to the above and by induction, we can show that there exists a constant $K > 0$ such that

$$|y(x)| \leq K\varepsilon.$$

This completes the proof of our theorem. □

Chapter 2

Generlaized Hyers-Ulam Stability of Differential Equations

2.1 Generlaized Hyers-Ulam Stability of Linear Differential Equations

Throughout this section, $\mathbb{F}$ will denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{F}$ is said to be of exponential order if there are constants $A, B \in \mathbb{R}$ such that

$$|f(t)| \le Ae^{tB}$$

for all $t > 0$. For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order, we define the Laplace transform of f by

$$F(s) = \int_0^\infty f(t)e^{-st}dt.$$

There exists a unique number $-\infty \le \sigma < \infty$ such that this integral converges if $\Re(s) > \sigma$ and diverges if $\Re(s) < \sigma$, where $\Re(s)$ denotes the real part of the (complex) number s. The number σ is called the abscissa of convergence and denoted by σ_f. It is well known that $|F(s)| \to 0$ as $\Re(s) \to \infty$. Furthermore, f is analytic on the open right half plane $\{s \in \mathbb{C} : \Re(s) > \sigma\}$ and we have

$$\frac{d}{ds}F(s) = -\int_0^\infty te^{-st}f(t)dt \quad (\Re(s) > \sigma).$$

The Laplace transform of f is sometimes denoted by $\mathcal{L}(f)$. It is well known that $\mathcal{L}$ is linear and one-to-one.

Conversely, let $f(t)$ be a continuous function whose Laplace transform $F(s)$ has the abscissa of convergence σ_f, then the formula for the inverse Laplace transforms yields

$$f(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+iy)t}F(\alpha+iy)dy$$

for any real constant $\alpha > \sigma_f$, where the first integral is taken along the vertical line $\Re(s) = \alpha$ and converges as an improper Riemann integral and the second integral is used as an alternative notation for the first integral (see [1]). Hence, we have

$$\mathcal{L}(f)(s) = \int_0^{\infty} f(t)e^{-st}dt \quad (\Re(s) > \sigma_f)$$
$$\mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+iy)t}F(\alpha+iy)dy \quad (\alpha > \sigma_f).$$

The convolution of two integrable functions $f, g : (0,\infty) \to \mathbb{F}$ is defined by

$$(f * g)(t) := \int_0^t f(t-x)g(x)dx.$$

Then $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$.

Lemma 2.1.1. *[12] Let* $P(s) = \sum_{k=0}^{n} \alpha_k s^k$ *and* $Q(s) = \sum_{k=0}^{m} \beta_k s^k$*, where* m, n *are nonnegative integers with* $m < n$ *and* α_k, β_k *are scalars. Then there exists an infinitely differentiable function* $g : (0,\infty) \to \mathbb{F}$ *such that*

$$\mathcal{L}(g) = \frac{Q(s)}{P(s)} \quad (\Re(s) > \sigma_P)$$

and

$$g^{(i)}(0) = \begin{cases} 0 & (\text{for } i \in \{0,1,\dots,n-m-2\}), \\ \beta_m/\alpha_n & (\text{for } i = n-m-1) \end{cases}$$

where $\sigma_P = \max\{\Re(s) : P(s) = 0\}$.

Lemma 2.1.2. *[12] Given an integer* $n > 1$*, let* $f : (0,\infty) \to \mathbb{F}$ *be a continuous function and let* $P(s)$ *be a complex polynomial of degree* n*. Then there exists an* n *times continuously differentiable function* $h : (0,\infty) \to \mathbb{F}$ *such that*

$$\mathcal{L}(h) = \frac{\mathcal{L}(f)}{P(s)} \quad (\Re(s) > \max\{\sigma_P, \sigma_f\}),$$

where $\sigma_P = \max\{\Re(s) : P(s) = 0\}$ *and* σ_f *is the abscissa of convergence for* f*. In particular, it holds that* $h^{(i)}(0) = 0$ *for every* $i \in \{0,1,\dots,n-1\}$.

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. In the following theorem, using the Laplace transform method, we investigate the generalized Hyers-Ulam stability of the linear differential equation of first order

$$y'(t) + \alpha y(t) = f(t). \tag{2.1.1}$$

Theorem 2.1.3. *[30] Let α be a constant in $\mathbb{F}$ and let $\varphi : (0, \infty) \to (0, \infty)$ be an integrable function. If a continuously differentiable function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality*

$$|y'(t) + \alpha y(t) - f(t)| \leq \varphi(t) \tag{2.1.2}$$

for all $t > 0$, then there exists a solution $y_\alpha : (0, \infty) \to \mathbb{F}$ of the differential equation (2.1.1) *such that*

$$|y(t) - y_\alpha(t)| \leq e^{-\Re(\alpha)t} \int_0^t e^{\Re(\alpha)x} \varphi(x)dx$$

for any $t > 0$.

Proof. If we define a function $z : (0, \infty) \to \mathbb{F}$ by $z(t) := y'(t) + \alpha y(t) - f(t)$ for each $t > 0$, then

$$\mathcal{L}(y) - \frac{y(0) + \mathcal{L}(f)}{s + \alpha} = \frac{\mathcal{L}(z)}{s + \alpha}. \tag{2.1.3}$$

If we set $y_\alpha(t) := y(0)e^{-\alpha t} + (E_{-\alpha} * f)(t)$, where $E_{-\alpha}(t) = e^{-\alpha t}$, then $y_\alpha(0) = y(0)$ and

$$\mathcal{L}(y_\alpha) = \frac{y(0) + \mathcal{L}(f)}{s + \alpha} = \frac{y_\alpha(0) + \mathcal{L}(f)}{s + \alpha}. \tag{2.1.4}$$

Hence, we get

$$\mathcal{L}\big(y'_\alpha(t) + \alpha y_\alpha(t)\big) = s\mathcal{L}(y_\alpha) - y_\alpha(0) + \alpha \mathcal{L}(y_\alpha) = \mathcal{L}(f).$$

Since $\mathcal{L}$ is a one-to-one operator, it holds that

$$y'_\alpha(t) + \alpha y_\alpha(t) = f(t).$$

Thus, y_α is a solution of (2.1.1).
Moreover, by (2.1.3) and (2.1.4), we obtain $\mathcal{L}(y) - \mathcal{L}(y_\alpha) = \mathcal{L}(E_{-\alpha} * z)$. Therefore, we have

$$y(t) - y_\alpha(t) = (E_{-\alpha} * z)(t). \tag{2.1.5}$$

In view of (2.1.2), it holds that

$$|z(t)| \leq \varphi(t) \tag{2.1.6}$$

for all $t > 0$, and it follows from the definition of convolution, (2.1.5), and (2.1.6) that

$$\begin{aligned}|y(t) - y_\alpha(t)| &= |(E_{-\alpha} * z)(t)| \\ &= \left| \int_0^t E_{-\alpha}(t-x)z(x)dx \right| \\ &\leq \int_0^t \left| e^{-\alpha(t-x)} \right| \varphi(x)dx \\ &\leq e^{-\Re(\alpha)t} \int_0^t e^{\Re(\alpha)x}\varphi(x)dx\end{aligned}$$

for all $t > 0$. (We remark that $\int_0^t e^{\Re(\alpha)x}\varphi(x)dx$ exists for each $t > 0$ provided φ is an integrable function.) □

Corollary 2.1.4. *[30] Let α be a constant in $\mathbb{F}$ and let $\varphi : (0,\infty) \to (0,\infty)$ be an integrable function such that*

$$\int_0^t e^{\Re(\alpha)(x-t)}\varphi(x)dx \leq K\varphi(t) \tag{2.1.7}$$

for all $t > 0$ and for some positive real constant K. If a continuously differentiable function $y : (0,\infty) \to \mathbb{F}$ satisfies the inequality (2.1.2) *for all $t > 0$, then there exists a solution $y_\alpha : (0,\infty) \to \mathbb{F}$ of the differential equation* (2.1.1) *such that*

$$|y(t) - y_\alpha(t)| \leq K\varphi(t)$$

for any $t > 0$.

In the following remark, we show that there exists an integrable function $\varphi : (0,\infty) \to (0,\infty)$ satisfying the condition (2.1.7).

Remark 2.1.5. *[30] Let α be a constant in $\mathbb{F}$ with $\Re(\alpha) > -1$. If we define $\varphi(t) = Ae^t$ for all $t > 0$ and for some $A > 0$, then we have*

$$\begin{aligned}\int_0^t e^{\Re(\alpha)(x-t)}\varphi(x)dx &= \int_0^t e^{\Re(\alpha)(x-t)}Ae^x dx \\ &= \frac{1}{1+\Re(\alpha)}\left(Ae^t - Ae^{-\Re(\alpha)t}\right) \\ &\leq \frac{1}{1+\Re(\alpha)}\varphi(t)\end{aligned}$$

for each $t > 0$.

Now, we apply the Laplace transform method to the proof of the generalized Hyers-Ulam stability of the linear differential equation of second order

$$y''(t) + \beta y'(t) + \alpha y(t) = f(t). \tag{2.1.8}$$

Theorem 2.1.6. *[30] Let α and β be constants in $\mathbb{F}$ such that there exist $a, b \in \mathbb{F}$ with $a + b = -\beta$, $ab = \alpha$, and $a \neq b$. Assume that $\varphi : (0, \infty) \to (0, \infty)$ is an integrable function. If a twice continuously differentiable function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality*

$$|y''(t) + \beta y'(t) + \alpha y(t) - f(t)| \leq \varphi(t) \tag{2.1.9}$$

for all $t > 0$, then there exists a solution $y_c : (0, \infty) \to \mathbb{F}$ of the differential equation (2.1.8) *such that*

$$|y(t) - y_c(t)| \leq \frac{e^{\Re(a)t}}{|a-b|} \int_0^t e^{-\Re(a)x} \varphi(x) dx + \frac{e^{\Re(b)t}}{|a-b|} \int_0^t e^{-\Re(b)x} \varphi(x) dx$$

for all $t > 0$.

Proof. If we define a function $z : (0, \infty) \to \mathbb{F}$ by $z(t) := y''(t) + \beta y'(t) + \alpha y(t) - f(t)$ for each $t > 0$, then we have

$$\mathcal{L}(z) = \big(s^2 + \beta s + \alpha\big)\mathcal{L}(y) - [sy(0) + \beta y(0) + y'(0)] - \mathcal{L}(f). \tag{2.1.10}$$

In view of (2.1.10), a function $y_0 : (0, \infty) \to \mathbb{F}$ is a solution of (2.1.8) if and only if

$$\big(s^2 + \beta s + \alpha\big)\mathcal{L}(y_0) - sy_0(0) - [\beta y_0(0) + y_0'(0)] = \mathcal{L}(f). \tag{2.1.11}$$

Now, since $s^2 + \beta s + \alpha = (s - a)(s - b)$, (2.1.10) implies that

$$\mathcal{L}(y) - \frac{sy(0) + [\beta y(0) + y'(0)] + \mathcal{L}(f)}{(s-a)(s-b)} = \frac{\mathcal{L}(z)}{(s-a)(s-b)}. \tag{2.1.12}$$

If we set

$$y_c(t) := y(0)\frac{ae^{at} - be^{bt}}{a-b} + [\beta y(0) + y'(0)]E_{a,b}(t) + (E_{a,b} * f)(t), \tag{2.1.13}$$

where $E_{a,b}(t) := \frac{e^{at} - e^{bt}}{a-b}$, then $y_c(0) = y(0)$. Moreover, since

$$y_c'(t) = y(0)\frac{a^2e^{at} - b^2e^{bt}}{a-b} + [\beta y(0) + y'(0)]\frac{ae^{at} - be^{bt}}{a-b} + \frac{d}{dt}(E_{a,b} * f)(t),$$

$$(E_{a,b} * f)(t) = \frac{e^{at}}{a-b}\int_0^t e^{-ax} f(x) dx - \frac{e^{bt}}{a-b}\int_0^t e^{-bx} f(x) dx,$$

we have

$$\begin{aligned} y_c'(0) &= y(0)\frac{a^2-b^2}{a-b} + [\beta y(0) + y'(0)]\frac{a-b}{a-b} \\ &= (a+b)y(0) + \beta y(0) + y'(0) \\ &= y'(0). \end{aligned}$$

It follows from (2.1.13) that

$$\mathcal{L}(y_c) = \frac{s y_c(0) + [\beta y_c(0) + y_c'(0)] + \mathcal{L}(f)}{(s-a)(s-b)}. \tag{2.1.14}$$

Now, (2.1.11) and (2.1.14) imply that y_c is a solution of (2.1.8). Applying (2.1.12) and (2.1.14) and considering the facts that
$y_c(0) = y(0)$, $y_c'(0) = y'(0)$, and $\mathcal{L}(E_{a,b} * z) = \frac{\mathcal{L}(z)}{(s-a)(s-b)}$, we obtain
$\mathcal{L}(y) - \mathcal{L}(y_c) = \mathcal{L}(E_{a,b} * z)$ or equivalently, $y(t) - y_c(t) = (E_{a,b} * z)(t)$.
In view of (2.1.9), it holds that $|z(t)| \leq \varphi(t)$, and it follows from the definition of the convolution that

$$\begin{aligned} |y(t) - y_c(t)| &= |(E_{a,b} * z)(t)| \\ &\leq \frac{e^{\Re(a)t}}{|a-b|}\int_0^t e^{-\Re(a)x}\varphi(x)dx + \frac{e^{\Re(b)t}}{|a-b|}\int_0^t e^{-\Re(b)x}\varphi(x)dx \end{aligned}$$

for any $t > 0$. We remark that $\int_0^t e^{-\Re(a)x}\varphi(x)dx$ and $\int_0^t e^{-\Re(b)x}\varphi(x)dx$ exist for any $t > 0$ provided φ is an integrable function. □

Corollary 2.1.7. *[30] Let α and β be constants in $\mathbb{F}$ such that there exist $a, b \in \mathbb{F}$ with $a + b = -\beta$, $ab = \alpha$, and $a \neq b$. Assume that $\varphi : (0,\infty) \to (0,\infty)$ is an integrable function for which there exists a positive real constant K with*

$$\int_0^t \left(e^{\Re(a)(t-x)} + e^{\Re(b)(t-x)}\right)\varphi(x)dx \leq K\varphi(t) \tag{2.1.15}$$

for all $t > 0$. If a twice continuously differentiable function $y : (0,\infty) \to \mathbb{F}$ satisfies the inequality (2.1.9) *for all $t > 0$, then there exists a solution $y_c : (0,\infty) \to \mathbb{F}$ of the differential equation* (2.1.8) *such that*

$$|y(t) - y_c(t)| \leq \frac{K}{|a-b|}\varphi(t)$$

for all $t > 0$.

We now show that there exists an integrable function $\varphi : (0, \infty) \to (0, \infty)$ which satisfies the condition (2.1.15).

Remark 2.1.8. *[30] Let α and β be constants in $\mathbb{F}$ such that there exist $a, b \in \mathbb{F}$ with $a + b = -\beta$, $ab = \alpha$, $\Re(a) < 1$, $\Re(b) < 1$, and $a \neq b$. If we define $\varphi(t) = Ae^t$ for all $t > 0$ and for some $A > 0$, then we get*

$$\begin{aligned}
&\int_0^t \left(e^{\Re(a)(t-x)} + e^{\Re(b)(t-x)}\right)\varphi(x)dx \\
&\qquad = \int_0^t \left(e^{\Re(a)(t-x)} + e^{\Re(b)(t-x)}\right)Ae^x dx \\
&\qquad = \frac{A}{1-\Re(a)}\left(e^t - e^{\Re(a)t}\right) + \frac{A}{1-\Re(b)}\left(e^t - e^{\Re(b)t}\right) \\
&\qquad \leq \left(\frac{1}{1-\Re(a)} + \frac{1}{1-\Re(b)}\right)\varphi(t)
\end{aligned}$$

for all $t > 0$.

Similarly, we apply the Laplace transform method to investigate the generalized Hyers-Ulam stability of the linear differential equation of nth order

$$y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) = f(t) \tag{2.1.16}$$

Theorem 2.1.9. *[30] Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be scalars in $\mathbb{F}$ with $\alpha_n = 1$, where n is an integer larger than 1. Assume that $\varphi : (0, \infty) \to (0, \infty)$ is an integrable function of exponential order. If an n times continuously differentiable function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality*

$$\left| y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) - f(t) \right| \leq \varphi(t) \tag{2.1.17}$$

for all $t > 0$, then there exist real constants $M > 0$ and σ_g and a solution $y_c : (0, \infty) \to \mathbb{F}$ of the differential equation (2.1.16) such that

$$|y(t) - y_c(t)| \leq M \int_0^t e^{\alpha(t-x)}\varphi(x)dx$$

for all $t > 0$ and $\alpha > \sigma_g$.

Proof. Applying the integration by parts repeatedly, we derive

$$\mathcal{L}\big(y^{(k)}\big) = s^k \mathcal{L}(y) - \sum_{j=1}^{k} s^{k-j} y^{(j-1)}(0)$$

for any integer $k > 0$. Using this formula, we can prove that a function $y_0 : (0, \infty) \to \mathbb{F}$ is a solution of (2.1.16) if and only if

$$\begin{aligned}
\mathcal{L}(f) &= \sum_{k=0}^{n} \alpha_k s^k \mathcal{L}(y_0) - \sum_{k=1}^{n} \alpha_k \sum_{j=1}^{k} s^{k-j} y_0^{(j-1)}(0) \\
&= \sum_{k=0}^{n} \alpha_k s^k \mathcal{L}(y_0) - \sum_{j=1}^{n} \sum_{k=j}^{n} \alpha_k s^{k-j} y_0^{(j-1)}(0) \\
&= P_{n,0}(s)\mathcal{L}(y_0) - \sum_{j=1}^{n} P_{n,j}(s) y_0^{(j-1)}(0), \qquad (2.1.18)
\end{aligned}$$

where $P_{n,j}(s) := \sum\limits_{k=j}^{n} \alpha_k s^{k-j}$ for $j \in \{0, 1, \ldots, n\}$.
Let us define a function $z : (0, \infty) \to \mathbb{F}$ by

$$z(t) := y^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k y^{(k)}(t) - f(t) \qquad (2.1.19)$$

for all $t > 0$. Then, similarly as in (2.1.18), we obtain

$$\mathcal{L}(z) = P_{n,0}(s)\mathcal{L}(y) - \sum_{j=1}^{n} P_{n,j}(s) y^{(j-1)}(0) - \mathcal{L}(f).$$

Hence, we get

$$\mathcal{L}(y) - \frac{1}{P_{n,0}(s)} \left(\sum_{j=1}^{n} P_{n,j}(s) y^{(j-1)}(0) + \mathcal{L}(f) \right) = \frac{\mathcal{L}(z)}{P_{n,0}(s)}. \qquad (2.1.20)$$

Let σ_f be the abscissa of convergence for f, let $s_1, s_2, \ldots, s_n$ be the roots of the polynomial $P_{n,0}(s)$, and let $\sigma_P = \max\{\Re(s_k) : k \in \{1, 2, \ldots, n\}\}$. For any s with $\Re(s) > \max\{\sigma_f, \sigma_P\}$, we set

$$G(s) := \frac{1}{P_{n,0}(s)} \left(\sum_{j=1}^{n} P_{n,j}(s) y^{(j-1)}(0) + \mathcal{L}(f) \right). \qquad (2.1.21)$$

By Lemma 2.1.2, there exists an n times continuously differentiable function f_0 such that

$$\mathcal{L}(f_0) = \frac{\mathcal{L}(f)}{P_{n,0}(s)} \tag{2.1.22}$$

for all s with $\Re(s) > \max\{\sigma_f, \sigma_P\}$ and

$$f_0^{(i)}(0) = 0 \tag{2.1.23}$$

for any $i \in \{0, 1, \ldots, n-1\}$.
For $j \in \{1, 2, \ldots, n\}$, we note that

$$\frac{P_{n,j}(s)}{P_{n,0}(s)} = \frac{1}{s^j} - \frac{\sum_{k=0}^{j-1} \alpha_k s^k}{s^j P_{n,0}(s)} \tag{2.1.24}$$

for every s with $\Re(s) > \max\{0, \sigma_P\}$. Applying Lemma 2.1.1 for the case of $Q(s) = \sum_{k=0}^{j-1} \alpha_k s^k$ and $P(s) = s^j P_{n,0}(s)$, we can find an infinitely differentiable function g_j such that

$$\mathcal{L}(g_j) = \frac{\sum_{k=0}^{j-1} \alpha_k s^k}{s^j P_{n,0}(s)} \tag{2.1.25}$$

and $g_j^{(k)}(0) = 0$ for $k \in \{0, 1, \ldots, n-1\}$.
Let

$$f_j(t) := \frac{t^{j-1}}{(j-1)!} - g_j(t) \tag{2.1.26}$$

for $j \in \{1, 2, \ldots, n\}$. Then we have

$$f_j^{(i)}(0) = \begin{cases} 0 & (\text{for } i \in \{0, 1, \ldots, j-2, j, j+1, \ldots, n-1\}), \\ 1 & (\text{for } i = j-1). \end{cases} \tag{2.1.27}$$

If we define

$$y_c(t) := \sum_{j=1}^{n} y^{(j-1)}(0) f_j(t) + f_0(t),$$

then the conditions (2.1.23) and (2.1.27) imply that

$$y_c^{(i)}(0) = y^{(i)}(0) \tag{2.1.28}$$

for every $i \in \{0, 1, \ldots, n-1\}$. Moreover, it follows from (2.1.21) to (2.1.28) that

$$\begin{aligned}\mathcal{L}(y_c) &= \sum_{j=1}^{n} y^{(j-1)}(0)\mathcal{L}(f_j) + \mathcal{L}(f_0) \\ &= \sum_{j=1}^{n} y^{(j-1)}(0)\left(\frac{1}{s^j} - \mathcal{L}(g_j)\right) + \frac{\mathcal{L}(f)}{P_{n,0}(s)} \\ &= \frac{1}{P_{n,0}(s)}\left(\sum_{j=1}^{n} P_{n,j}(s)y^{(j-1)}(0) + \mathcal{L}(f)\right) \qquad (2.1.29)\end{aligned}$$

for each s with $\Re(s) > \max\{0, \sigma_f, \sigma_p\}$.
Now, (2.1.18) implies that y_c is a solution of (2.1.16). Moreover, by (2.1.20) and (2.1.29), we have

$$\mathcal{L}(y) - \mathcal{L}(y_c) = \frac{\mathcal{L}(z)}{P_{n,0}(s)}. \qquad (2.1.30)$$

Applying Lemma 2.1.1 for the case of $Q(s) = 1$ and $P(s) = P_{n,0}(s)$, we find an infinitely differentiable function $g : (0, \infty) \to \mathbb{F}$ such that

$$\mathcal{L}(g) = \frac{1}{P_{n,0}(s)} \qquad (2.1.31)$$

which implies that

$$g(t) = \mathcal{L}^{-1}\left(\frac{1}{P_{n,0}(s)}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(\alpha+iy)t}\frac{1}{P_{n,0}(\alpha+iy)}dy$$

for any real constant $\alpha > \sigma_g$.
Moreover, it holds that

$$\begin{aligned}|g(t-x)| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|e^{(\alpha+iy)(t-x)}\right| \frac{1}{|P_{n,0}(\alpha+iy)|}dy \\ &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\alpha(t-x)}\frac{1}{|P_{n,0}(\alpha+iy)|}dy \\ &\le \frac{1}{2\pi}e^{\alpha(t-x)}\int_{-\infty}^{\infty}\frac{1}{|P_{n,0}(\alpha+iy)|}dy \\ &\le Me^{\alpha(t-x)} \qquad (2.1.32)\end{aligned}$$

for all $\alpha > \sigma_g$, where

$$M = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|P_{n,0}(\alpha+iy)|}dy < \infty,$$

because n is an integer larger than 1. By (2.1.17) and (2.1.19), it also holds that $|z(t)| \leq \varphi(t)$ for all $t > 0$.
In view of (2.1.30), (2.1.31), and (2.1.32), we get

$$\mathcal{L}(y) - \mathcal{L}(y_c) = \mathcal{L}(g)\mathcal{L}(z) = \mathcal{L}(g * z).$$

Consequently, we have $y(t) - y_c(t) = (g * z)(t)$ for any $t > 0$. Hence, it follows from (2.1.17), (2.1.19), and (2.1.32) that

$$|y(t) - y_c(t)| = |(g * z)(t)| \leq \int_0^t |g(t-x)||z(x)|dx \leq M \int_0^t e^{\alpha(t-x)}\varphi(x)dx$$

for all $t > 0$ and for any real constant $\alpha > \sigma_g$, which completes the proof. □

Corollary 2.1.10. *[30] Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be scalars in $\mathbb{F}$ with $\alpha_n = 1$, where n is an integer larger than 1. Assume that there exist real constants α and $K > 0$ such that a function $\varphi : (0, \infty) \to (0, \infty)$ satisfies*

$$\int_0^t e^{\alpha(t-x)}\varphi(x)dx \leq K\varphi(t)$$

for all $t > 0$. Moreover, assume that the constant σ_g given in Theorem 2.1.9 is less than α. If an n times continuously differentiable function $y : (0, \infty) \to \mathbb{F}$ satisfies the inequality (2.1.17) for all $t > 0$, then there exist a real constants $M > 0$ and a solution $y_c : (0, \infty) \to \mathbb{F}$ of the differential equation (2.1.16) such that

$$|y(t) - y_c(t)| \leq KM\varphi(t)$$

for all $t > 0$.

Remark 2.1.11. *[30] Assume that $\alpha < 1$. If we define $\varphi(t) = Ae^t$ for all $t > 0$ and for some $A > 0$, then we get*

$$\int_0^t e^{\alpha(t-x)}\varphi(x)dx = \int_0^t e^{\alpha(t-x)}Ae^x dx = \frac{A}{1-\alpha}\left(e^t - e^{\alpha t}\right) \leq \frac{1}{1-\alpha}\varphi(t)$$

for all $t > 0$.

2.2 Generlaized Hyers-Ulam Stability of Differential Equations with boundary Conditions

2.2.1 Generalized Superstability of Differential Equations with Initial Conditions

In this subsection, we investigate the generalized superstability of linear differential equation of nth-order in the form of

$$y^{(n)}(x) + \beta(x)y(x) = 0, \tag{2.2.1}$$

with initial conditions

$$y(a) = y^{'}(a) = \cdots = y^{(n-1)}(a) = 0, \tag{2.2.2}$$

where $n \in \mathbb{N}^{+}$, $y \in C^{n}[a,b], \beta \in C^{0}[a,b],\ -\infty < a < b < +\infty$.
In addition to that we investigate the generalized superstability of differential equations of second order in the form of $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ and the superstability of linear differential equations with constant coefficients.

First of all, we give the definition of generalized superstability with initial and boundary conditions.

Definition 2.2.1. *[19],[25] Assume that for any function $y \in C^{n}[a,b]$, if y satisfies the differential inequality*

$$\left|\varphi\big(f, y, y', \ldots, y^{(n)}\big)\right| \leq \varphi(x)$$

for all $x \in [a,b]$ and for some function $\varphi : [a,b] \to [0,\infty)$ with initial(or boundary) conditions, then either y is a solution of the differential equation

$$\varphi\big(f, y, y', \ldots, y^{(n)}\big) = 0 \tag{2.2.3}$$

or $|y(x)| \leq \Phi(x)$ for any $x \in [a,b]$, where $\Phi : I \to [0,\infty)$ is a function not depending on y explicitly. Then, we say that Eq.(2.2.3) has generalized superstability with initial(or boundary) conditions.

In this subsection, given the closed interval $I = [a,b]$, we assume that $\varphi : I \to [0,\infty)$ and let $\mathbf{M}\left(p(x)\right)$ denote $\max_{\tau \in [a,x]} |p(\tau)|$ for every $p \in C(I,\mathbb{R})$.

Theorem 2.2.2. *[19] If $|\beta(x)| < n!/(b-a)^n$ for every $x \in I$, then Eq.(2.2.1) has generalized superstability with initial conditions (2.2.2).*

Proof. Suppose that a function $y \in C^n(I, \mathbb{R})$ satisfies the inequality

$$\left|y^{(n)}(x) + \beta(x)y(x)\right| \le \varphi(x), \tag{2.2.4}$$

for all $x \in I$,
By Taylor formula, we have

$$y(x) = y(a) + y'(a)(x-a) + \cdots + \frac{y^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{y^{(n)}(\xi)}{n!}(x-a)^n.$$

Therefore,

$$|y(x)| = \left|\frac{y^{(n)}(\xi)}{n!}(x-a)^n\right| \le \mathbf{M}(y^{(n)}(x))\frac{(x-a)^n}{n!}$$

for every $x \in [a, b]$. Then,

$$\begin{aligned}\mathbf{M}\left(y(x)\right) &\le \mathbf{M}\left(\mathbf{M}(y^{(n)}(x))\frac{(x-a)^n}{n!}\right)\\ &\le \mathbf{M}\left(\mathbf{M}(y^{(n)}(x))\right)\mathbf{M}\left(\frac{(x-a)^n}{n!}\right)\\ &= \mathbf{M}(y^{(n)}(x))\frac{(x-a)^n}{n!}\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{M}(y(x)) &\le \mathbf{M}(y^{(n)}(x))\frac{(x-a)^n}{n!}\\ &\le \frac{(x-a)^n}{n!}\mathbf{M}\left(y^{(n)}(x) + \beta(x)y(x)\right) + \frac{(x-a)^n}{n!}\mathbf{M}\,|\beta(x)|\,\mathbf{M}(y(x))\\ &\le \frac{(x-a)^n}{n!}\mathbf{M}\left(\varphi(x)\right) + \frac{(b-a)^n}{n!}\max|\beta(x)|\mathbf{M}\left(y(x)\right).\end{aligned}$$

Let $C_1 = 1 - \frac{(b-a)^n}{n!}\max|\beta(x)|$. It easy to see that

$$\mathbf{M}\left(y(x)\right) \le \frac{(x-a)^n}{n!C_1}\mathbf{M}\left(\varphi(x)\right).$$

Moreover, $|y(x)| \le \mathbf{M}\left(y(x)\right)$, which completes the proof of our theorem. □

In the following theorem, we investigate the generalized superstability of the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \tag{2.2.5}$$

with initial conditions

$$y(a) = 0 = y^{'}(a), \tag{2.2.6}$$

where $y \in C^2[a,b],\ p \in C[a,b],\ q \in C^0[a,b],\ -\infty < a < b < +\infty$.

Theorem 2.2.3. *[19] If* $\max\{q(x) - \frac{1}{2}p^{'}(x) - \frac{p^2(x)}{4}\} < 2/(b-a)^2$*, then (2.2.5) has generalized superstability with initial conditions (2.2.6).*

Proof. Suppose that $y \in C^2[a,b]$ satisfies the inequality

$$\left| y^{''}(x) + p(x)y^{'}(x) + q(x)y(x) \right| \le \varphi(x). \tag{2.2.7}$$

Let

$$u(x) = y^{''}(x) + p(x)y^{'}(x) + q(x)y(x), \tag{2.2.8}$$

for all $x \in [a,b]$, and define $z(x)$ by

$$y(x) = z(x)exp(-\frac{1}{2}\int_a^x p(\tau)d\tau). \tag{2.2.9}$$

By a substitution (2.2.9) in (2.2.8), we obtain

$$z^{''}(x) + \big(q(x) - \frac{1}{2}p^{'}(x) - \frac{p^2(x)}{4}\big)z(x) = u(x)exp(\frac{1}{2}\int_a^x p(\tau)d\tau).$$

Then it follows from inequality (2.2.7) that

$$\begin{aligned} \left| z^{''}(x) + \big(q(x) - \frac{1}{2}p^{'}(x) - \frac{p^2(x)}{4}\big)z(x) \right| &= |u(x)|\, exp(\frac{1}{2}\int_a^x p(\tau)d\tau) \\ &\le \varphi(x)exp(\frac{1}{2}\int_a^x p(\tau)d\tau) \end{aligned}$$

From (2.2.6) and (2.2.9) we have

$$z(a) = 0 = z(b). \tag{2.2.10}$$

It follows from Theorem 2.2.2 that there exists a constant $C_1 > 0$ such that

$$|z(x)| \le \frac{(x-a)^n}{n!C_1}\mathbf{M}\left(\varphi(x)exp(\frac{1}{2}\int_a^x p(\tau)d\tau) \right).$$

From (2.2.9) we have

$$|y(x)| \leq \frac{(x-a)^n}{n!C_1}\mathbf{M}\left(\varphi(x)exp(\frac{1}{2}\int_a^x p(\tau)d\tau)\right)exp(-\frac{1}{2}\int_a^x p(\tau)d\tau).$$

Thus (2.2.5) has generalized superstability with initial conditions (2.2.6). □

2.2.2 Hyers-Ulam-Rassias Stability of Linear Differential Equations with Boundary Conditions

Lemma 2.2.4. *[24] Let $y \in C^n[a,b]$ and satisfies the initial conditions*

$$y(a) = y^{'}(a) = \cdots = y^{(n-1)}(a) = 0, \tag{2.2.11}$$

then $\max|y(x)| \leq \frac{(b-a)^n}{n!}\max\left|y^{(n)}(x)\right|$.

Proof. By Taylor formula, we have

$$y(x) = y(a) + y'(a)(x-a) + \cdots + \frac{y^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \frac{y^{(n)}(\xi)}{n!}(x-a)^n.$$

We have $(x-a)^n \leq (b-a)^n$. Therefore,

$$|y(x)| = \left|\frac{y^{(n)}(\xi)}{n!}(x-a)^n\right| \leq \max\left|y^{(n)}(x)\right|\frac{(b-a)^n}{n!}$$

for every $x \in [a,b]$. □

In the following theorems, we prove the Hyers-Ulam-Rassias stability of the following linear differential equation

$$y^{''}(x) + \beta(x)y(x) = 0 \tag{2.2.12}$$

with boundary conditions

$$y(a) = 0 = y(b) \tag{2.2.13}$$

or with initial conditions

$$y(a) = 0 = y^{'}(a) \tag{2.2.14}$$

where $I = [a,b], y \in C^2(I,\mathbb{R}),\ \beta \in C(I,\mathbb{R}),\ -\infty < a < b < +\infty$.

Given the closed interval I and a function $\beta : I \to \mathbb{R}$, define a function $f : I \to \mathbb{R}$ by

$$f(x) = y^{''}(x) + \beta(x)y(x), \tag{2.2.15}$$

for all $x \in I$.

Theorem 2.2.5. *[24] Given the closed interval I and a function $\beta : I \to \mathbb{R}$, assume that $\varphi : I \to [0, \infty)$ is a dicreasing (increasing) function and $max\,|\beta(x)| < 8/(b-a)^2$. If a function $y \in C^2(I, \mathbb{R})$ satisfies the inequality*

$$\left|y^{''}(x) + \beta(x)y(x)\right| \leq \varphi(x), \tag{2.2.16}$$

for all $x \in I$, with boundary conditions (2.2.13) such that the function (2.2.15) is increasing (decreasing) function then there exist a constant $K > 0$ and a solution $y_0 \in C^2(I, \mathbb{R})$ of the differential equation (2.2.12) with boundary conditions (2.2.13) such that

$$|y(x) - y_0(x)| \leq K\varphi(x) \tag{2.2.17}$$

for any $x \in I$.

Proof. We have by Lemma 1.3.1 that

$$max\,|y(x)| \leq \frac{(b-a)^2}{8} max \left|y^{''}(x)\right|.$$

Thus

$$\begin{aligned} max\,|y(x)| &\leq \frac{(b-a)^2}{8} max \left|y^{''}(x) + \beta(x)y(x)\right| + \frac{(b-a)^2}{8} max\,|\beta(x)|\, max\,|y(x)| \\ &\leq \frac{(b-a)^2}{8}\varphi(x) + \frac{(b-a)^2}{8} max\,|\beta(x)|\, max\,|y(x)|. \end{aligned}$$

Let $C = \frac{(b-a)^2}{8}$, $K = \frac{C}{1-C\ max|\beta(x)|}$. Obviously, $y_0(x) = 0$ is a solution of (2.2.12) with boundary conditions (2.2.13) and

$$|y(x) - y_0(x)| \leq K\ \varphi(x).$$

□

Theorem 2.2.6. *[24] Given the closed interval I and a function $\beta : I \to \mathbb{R}$, assume that $\varphi : I \to [0, \infty)$ is a decreasing (increasing) function and $max\,|\beta(x)| < 2/(b-a)^2$. If a function $y \in C^2(I, \mathbb{R})$ satisfies the inequality (2.2.16) for all $x \in I$, with initial conditions (2.2.14) such that the function (2.2.15) is increasing (decreasing) then there exist a solution $y_0 \in C^2(I, \mathbb{R})$ of the differential equation* (2.2.12) *and a constant $K > 0$ such that*

$$|y(x) - y_0(x)| \leq K\varphi(x) \tag{2.2.18}$$

for any $x \in I$.

Proof. We have by Lemma 1.3.2 that

$$max\,|y(x)| \leq \frac{(b-a)^2}{2} max \left| y^{''}(x) \right|.$$

Thus

$$\begin{aligned} max\,|y(x)| &\leq \frac{(b-a)^2}{2} max \left| y^{''}(x) + \beta(x)y(x) \right| + \frac{(b-a)^2}{2} max\,|\beta(x)|\, max\,|y(x)| \\ &\leq \frac{(b-a)^2}{2}\varphi(x) + \frac{(b-a)^2}{2} max\,|\beta(x)|\, max\,|y(x)|. \end{aligned}$$

Let $C = \frac{(b-a)^2}{2}$, $K = \frac{C}{1-C\ max|\beta(x)|}$. Obviously, $y_0(x) = 0$ is a solution of (2.2.12) with initial conditions (2.2.14) and $|y(x) - y_0(x)| \leq K\ \varphi(x)$. □
In the following theorems investigate the Hyers-Ulam-Rassias stability of linear differential equation of nth - order

$$y^{(n)}(x) + \beta(x)y(x) = 0, \tag{2.2.19}$$

with initial conditions (2.2.11).

Given the closed interval $I = [a, b]$ and a function $\beta : I \to \mathbb{R}$, define a function $f : I \to \mathbb{R}$ by

$$f(x) = y^n(x) + \beta(x)y(x), \tag{2.2.20}$$

for all $x \in I$.

Theorem 2.2.7. *[24] Given the closed interval I and a function $\beta : I \to \mathbb{R}$, assume that $\varphi : I \to [0, \infty)$ is a decreasing (increasing) function and $max\,|\beta(x)| < n!/(b-a)^n$. If a function $y_0 \in C^n[a, b]$ satisfies the inequality*

$$|y^n(x) + \beta(x)y(x)| \leq \varphi(x), \tag{2.2.21}$$

for all $x \in I$, with initial conditions (2.2.11) such that the function (2.2.20) is increasing (decreasing) then there exist a solution $y_0 \in C^n(I, \mathbb{R})$ of the differential equation (2.2.19) *and a constant $K > 0$ such that*

$$|y(x) - y_0(x)| \leq K\varphi(x) \tag{2.2.22}$$

for any $x \in I$.

Proof. We have by Lemma 2.2.4 that

$$max\,|y(x)| \leq \frac{(b-a)^n}{n!} max\,|y^n(x)|\,.$$

Thus

$$\begin{aligned} max\,|y(x)| &\leq \frac{(b-a)^n}{n!} max\,|y^n(x) + \beta(x)y(x)| + \frac{(b-a)^n}{n!} max\,|\beta(x)|\,max\,|y(x)| \\ &\leq \frac{(b-a)^n}{n!}\varphi(x) + \frac{(b-a)^n}{n!} max\,|\beta(x)|\,max\,|y(x)|\,. \end{aligned}$$

Let $C_1 = \frac{(b-a)^n}{n!}$, $K = \frac{C_1}{1-C_1\ max|\beta(x)|}$. Obviously, $y_0(x) = 0$ is a solution of (2.2.19) with initial conditions (2.2.11) and $|y(x) - y_0(x)| \leq K\ \varphi(x)$. □

Chapter 3

Hyers-Ulam Stability of System of Differential Equations

3.1 Hyers-Ulam Stability of Linear System of Differential Equations

In this section, by applying the fixed point alternative method, we give a necessary and sufficient condition in order that the first order linear system of differential equations $\dot{z}(t) + A(t)z(t) + B(t) = 0$ has the Hyers-Ulam-Rassias stability and find Hyers-Ulam stability constant under those conditions. In addition to that, we apply this result to a second order differential equation

$$\ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t) = 0.$$

Also, we apply to differential equations with constant coefficient in the same sense of proofs.

3.1.1 Preliminaries and Auxiliary Results

Definition 3.1.1. *([7],[2],[47]) Let I be any interval, let $z : I \to \mathbb{R}^n$, $A : I \to \mathbb{R}^{n\times n}$, $B : I \to \mathbb{R}^n$, then*

$$\dot{z}(t) + A(t)z(t) + B(t) = 0 \tag{3.1.1}$$

is Hyers-Ulam-Rassias stable with respect to $\varphi : I \to [0, \infty)$, with, $\|z(t)\| = \sum_{i=1}^{n} |z_i(t)|$, if there exists a real constant $K > 0$ such that for each solution $s \in$

$C^1(I, \mathbb{R}^n)$ *of inequality*

$$\|\dot{z}(t) + A(t)z(t) + B(t)\| \leq \psi(t)$$

there exists a solution $z \in C^1(I, \mathbb{R}^n)$ *of equation (3.1.1) with*

$$\|s(t) - z(t)\| \leq K\varphi(t)$$

, $\forall\ t\ \in\ I$.

Definition 3.1.2. *For a nonempty set* X, *a function* $d : X \times X \to [0, \infty]$ *is called a generalized metric on* X *if and only if* d *satisfies :*

(*i*) $d(x, y) = 0$ *if and only if* $x = y$;

(*ii*) $d(x, y) = d(y, x)$ *for all* $x,y \in X$;

(*iii*) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x,y,z \in X$.

Theorem 3.1.3. *(The fixed point alternative) [15] Let* (X, d) *be a generalized complete metric space. Assume that* $\Lambda : X \to X$ *is a strictly contractive operator with Lipschitz constant* $L < 1$. *If there exists a nonnegative integer* k *such that* $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ *for some* $x \in X$, *then the followings are true :*

(*a*) *The sequence* $\{\Lambda^n x\}$ *convergens to a fixed point* x^* *of* Λ;

(*b*) x^* *is the unique fixed point of* Λ *in*

$$X^* = \left\{y \in X/d(\Lambda^k x, y) < \infty\right\};$$

(*c*) *If* $y \in X^*$, *then* $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$.

Lemma 3.1.4. *[26] For given real numbers* a *and* b *with* $a < b$, *let* $I = [a, b]$ *be a closed interval and let be*
$X = \{f : I \to \mathbb{R}^n, is\ continuous\ function\}$, *and* $\mathrm{d} : \mathrm{X} \times \mathrm{X} \to [0,\infty]$ *a function is defined as follows :*

$$d(f, g) = \inf\{c \in [0, \infty]/\ \|f(t) - g(t)\| \leq c\varphi(t) \forall\ t \in I\}$$

where $\varphi(t) : I \to (0, \infty)$ *is a continuous function, then* d *is a generalized metric on* X.

Proof. By definition of a function d, then for all $f, g \in X$

(1) $d(f,g) = 0 \leftrightarrow f(t) = g(t)$ for all $t \in I$;

(2) $d(f,g) = d(g,f)$.

To prove that $d(f,g) \le d(f,h) + d(h,g)$ for all $f,g,h \in X$.
Assume that $d(f,g) > d(f,h) + d(h,g)$ for some $f,h,g \in X$. Then, by definition of d, there exists $t_0 \in I$ with

$$\begin{aligned} \|f(t_0) - g(t_0)\| &> \{d(f,h) - d(h,g)\}\,\varphi(t_0) \\ &= d(f,h)\varphi(t_0) - d(h,g)\varphi(t_0) \\ &\ge \|f(t_0) - h(t_0)\| + \|h(t_0) - g(t_0)\| \end{aligned}$$

and this is contradiction . □

Lemma 3.1.5. *[26] For given real numbers a and b with $a < b$, let $I = [a,b]$ be a closed interval and let*
$X = \{f : I \to \mathbb{R}^n$, is continuous function $\}$, consider a generalized metric function on X, d : X × X → [0,∞] *which is defined as follows:*

$$d(f,g) = \inf\{c \in [0,\infty] / \, \|f(t) - g(t)\| \le c\varphi(t) \ \forall \ t \in I\} \tag{3.1.2}$$

where $\varphi(t) : I \to (0,\infty)$ is a continuous function, then (X,d) is a complete metric space.

Proof. Let $\{h_n\}$ be a Cauchy sequence in (X,d). Then $\forall \varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ such that $d(h_m,h_n) \le \varepsilon \ \forall \ m,n \ge N_\varepsilon$. That means that with equation (3.1.2)

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} : \forall m,n \ge N_\varepsilon, \forall t \in I \ \|h_m(t) - h_n(t)\| \le \varepsilon\varphi(t). \tag{3.1.3}$$

If t is fixed, equation (3.1.3) implies that $\{h_n(t)\}$ is a cauchy sequence in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is complete, $\{h_n(t)\}$ converge for each $t \in I$. Thus, we can define a function $h : I \to \mathbb{R}^n$ by $h(t) = \lim\limits_{n\to\infty} h_n(t)$.
If we let $m \to \infty$, it then follows from (3.1.3) that

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} : \forall n \ge N_\varepsilon, \forall t \in I \ \|h(t) - h_n(t)\| \le \varepsilon\varphi(t), \tag{3.1.4}$$

that is, since φ is bounded on I, $\{h_n(t)\}$ converges uniformly to h. Hence, h is continuous and $h \in X$.
If we consider equation (3.1.2) and (3.1.4), then we may conclude that

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} : \forall n \ge N_\varepsilon \ d(h,h_n) \le \varepsilon$$

that is, the cauchy sequence $\{h_n(t)\}$ converge to h in (X,d). Hence, (X,d) is complete. □

3.1.2 Hyers-Ulam Stability of First Order System of Differential Equations

We will prove the Hyers-Ulam-Rassias stability for the equation (3.1.1) on the intervals $I = [a, b)$, where $-\infty < a < b \leq \infty$.

Theorem 3.1.6. *[26] Let $A : I \to \mathbb{R}^{n\times n}$ and $B : I \to \mathbb{R}^{n\times n}$ be continuous matrices functions and let for a positive constant N, such that $\|A(t)\| \geq N$ for all t in I. Assume that $\psi : I \to [o, \infty)$ is an integrable function with the property that there exists P in $(0, 1)$ such that*

$$\int_a^t \|A(t_1)\| \psi(t_1)dt_1 \leq P\psi(t) \tag{3.1.5}$$

for all $t \in I$. If a continuously differential function $z : I \to \mathbb{R}^{n\times n}$ verifies the relation :

$$\|\dot{z}(t) + A(t)z(t) + B(t)\| \leq \psi(t) \tag{3.1.6}$$

for all t in I. Then there exists a unique solution $s : I \to \mathbb{R}^{n\times n}$ of the equation (3.1.1) which verifies the following relation:

$$\|z(t) - s(t)\| \leq \frac{P}{N - NP}\psi(t) \tag{3.1.7}$$

for all $t \in I$ and $s(a) = z(a)$.

Proof. Let us consider the set

$$\Omega = \{h : I \to R^n/\text{h is continuous and h(a) = z(a)}\}$$

with a function $d : \Omega \times \Omega \to [0, \infty]$ defined on Ω as

$$\begin{aligned} d(h_1, h_2) &= d_\varphi(h_1, h_2) \\ &= \inf\{K > 0, \|h_1(t) - h_2(t)\| \leq K\varphi(t), \forall t \in I\} \end{aligned}$$

By lemma (3.1.4) and Lemma (3.1.5), the (Ω, d) is generalized complete metric space. We define the operator $T : \Omega \to \Omega$,

$$Th(t) = z(a) - \int_a^t (A(t_1)h(t_1) + B(t_1))dt_1, t \in I$$

for all $h \in \Omega$. Indeed Th is a continuously differentable function on I, since A and B are continuous functions and $Th(a) = z(a)$. Now, let $h_1, h_2 \in \Omega$. Then we have

$$\begin{aligned}
\|Th_1(t) - Th_2(t)\| &= \left\| \int_a^t A(t_1)(h_1(t_1) - h_2(t_1)dt_1) \right\| \\
&\leq \int_a^t \|A(t_1)\| \, \|h_1(t_1) - h_2(t_1)\| dt_1 \\
&\leq d(h_1, h_2) \int_a^t A(t_1)\psi(t_1)dt_1 \\
&\leq P\psi(t)d(h_1, h_2)\forall t \in I.
\end{aligned}$$

Therefore,

$$d(Th_1(t) - Th_2(t)) \leq Pd(h_1, h_2) \tag{3.1.8}$$

Thus, the operator T is a contraction with the constant P.
By integrating the both sides of the relation (3.1.6) on $[a, t]$ we obtain

$$\left\| z(t) - z(a) + \int_a^t (A(t_1)z(t_1) + B(t_1))dt_1 \right\| \leq \frac{P}{N}\psi(t) \; for \; all \; t \in I. \tag{3.1.9}$$

which means $d(z, Tz) \leq \frac{P}{N}\psi(t) < \infty$. By the fixed point alternative theorem (3.1.3) there exists an element $s = \lim_{n\to\infty} T^n z$ and s is a unique fixed point of T in the set $\Delta = \{h \in \Omega / d(T^{n_0}z, h) < \infty\}$.
It may be proved that $\Delta = \{h \in \Omega / d(z, h) < \infty\}$. Therefore, the set Δ is independent of n_0. To prove that the function s is a solution to the equation (3.1.1), we derive, with respect to t, the both sides of the following relation:

$$s(t) = Ts(t) \; \forall t \in I$$

Thus,

$$\dot{s}(t) = -A(t)s(t) - B(t)$$

for all $t \in I$, which implies that the function s is a solution of the equation (3.1.1) and verifies the relation $s(a) = z(a)$. Applying the fixed point alternative theorem again, we obtain $d(h, s) \leq \frac{1}{1-P}d(h, Th)$ for all $h \in \Delta$.
Since $z \in \Delta$, we have
$d(z, S) \leq \frac{1}{1-P}d(z, Tz) \leq \frac{P}{N(1-P)}$.
Hence, $\|z(t) - s(t)\| \leq \frac{P}{N(1-P)}\psi(t)$ for all $t \in I$.
This inequality proves the relation (3.1.1) . □

In the same manner it is possible to proved the following theorem of the Hyers-Ulam-Rassias stability of the equation (3.1.1) on the interval $J = (b, a]$, where $-\infty \leq b < a < \infty$.

Theorem 3.1.7. *[26] Let $A : J \to \mathbb{R}^{n\times n}$ and $B : J \to \mathbb{R}^{n\times n}$ be continuous matrices functions and let for a positive constant N, such that $\|A(t)\| \geq N$ for all $t \in J$.*
Assume that $\psi : J \to [o, \infty)$ is an integrable function with the property that there exists $P \in (0, 1)$such that

$$\int_a^t \|A(t_1)\| \, \psi(t_1) dt_1 \leq P\psi(t) \tag{3.1.10}$$

for all $t \in J$. If a continuously differential function $z : J \to \mathbb{R}^{n\times n}$ verifies the relation :

$$\|\dot{z}(t) + A(t)z(t) + B(t)\| \leq \psi(t) \tag{3.1.11}$$

for all $t \in J$, then there exists a unique solution $s : J \to \mathbb{R}^{n\times n}$ of the equation (3.1.1) which verifies the following relation:

$$\|z(t) - s(t)\| \leq \frac{P}{N - NP}\psi(t) \tag{3.1.12}$$

for all $t \in J$ and $s(a) = z(a)$.
The Hyers-Ulam-Rassias stability equation (3.1.1) on $\mathbb{R}$ will be proved by Theorem (3.1.6) and Theorem (3.1.7).

Corollary 3.1.8. *[26] Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R} \to \mathbb{R}^{n\times n}$ be continuous matrices functions and let for a positive constant N, such that $\|A(t)\| \geq N$ for all $t \in \mathbb{R}$. Assume that $\psi : \mathbb{R} \to [o, \infty)$ is an integrable function with the property that there exists $P \in (0, 1)$ such that*

$$\left| \int_0^t \|A(t_1)\| \, \psi(t_1) dt_1 \right| \leq P\psi(t) \tag{3.1.13}$$

for all $t \in \mathbb{R}$. If a continuously differential function $z : \mathbb{R} \to \mathbb{R}^{n\times n}$ verifies the relation :

$$\|\dot{z}(t) + A(t)z(t) + B(t)\| \leq \psi(t) \tag{3.1.14}$$

for all $t \in \mathbb{R}$, then there exists a unique solution $s : \mathbb{R} \to \mathbb{R}^{n\times n}$ of the equation (3.1.1) which verifies the following relation:

$$\|z(t) - s(t)\| \le \frac{P}{N - NP}\psi(t) \tag{3.1.15}$$

for all $t \in \mathbb{R}$ and $s(0) = z(0)$.

Proof. By relation (3.1.13) we have

$$\int_0^t \|A(t_1)\| \, \psi(t_1)dt_1 \le P\psi(t) \tag{3.1.16}$$

for all $t \ge 0$. Applying Theorem (3.1.6), there exists a solution of equation (3.1.1), $s_1 : [0,\infty) \to \mathbb{R}^{n\times n}$, which verifies relations (3.1.7) and $s_1(o) = z(o)$. From relation(3.1.13) we also obtain

$$\int_t^o \|A(t_1)\| \, \psi(t_1)dt_1 \le P\psi(t) \tag{3.1.17}$$

for all $t \le 0$. Applying Theorem (3.1.7), there exists a solution of equation (3.1.1), $s_2 : (-\infty, 0] \to \mathbb{R}^{n\times n}$ which verifies relation (3.1.12) and $s_2(o) = z(o)$. It is easy to check if the function

$$s(t) = \begin{cases} s_1(t), t \ge 0 \\ s_2(t), t < 0 \end{cases} \tag{3.1.18}$$

is a continuously differentiable function on $\mathbb{R}$, a solution of equation (3.1.1) on $\mathbb{R}$ and it verifies relation (3.1.15). ☐

Corollary 3.1.9. *[26] Let $A \ne 0$ be $n \times n$ constant matrix and $B : \mathbb{R} \to \mathbb{R}^{n\times n}$ be $n \times 1$ a continuous matrix function ($n \times 1$ constant matrix). Assume that $\psi : \mathbb{R} \to [o,\infty)$ is an integrable function with the property that there exists $P \in (0,1)$ such that*

$$\left| \int_0^t \psi(t_1) \right| \le \frac{P}{\|A\|}\psi(t) \tag{3.1.19}$$

for all $t \in \mathbb{R}$. If a continuously differential function $z : \mathbb{R} \to \mathbb{R}^{n\times n}$ verifies the relation :

$$\|\dot{z}(t) + Az(t) + B(t)\| \le \psi(t) \ (\|\dot{z}(t) + Az(t) + B\| \le \psi(t)) \tag{3.1.20}$$

for all $t \in \mathbb{R}$, *then there exists a unique solution* $s : \mathbb{R} \to \mathbb{R}^{n\times n}$ *of the equation*

$$\dot{z}(t) + Az(t) + B(t) = 0 \ (\dot{z}(t) + Az(t) + B = 0)$$

which verifies the following relation:

$$\|z(t) - s(t)\| \le \frac{P}{\|A\|(1-P)}\psi(t) \tag{3.1.21}$$

for all $t \in \mathbb{R}$ *and* $s(0) = z(0)$.

Proof. By Corollary (3.1.8) and let $N = \|A\|$. □

3.1.3 Hyers-Ulam Stability of Second Order Differential Equation

In this section we will prove the Hyers-Ulam-Rassias stability for the following scallar equation

$$\ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t) = 0 \tag{3.1.22}$$

In the same manner, at first we will prove the Hyers-Ulam-Rassias stability for the equation (3.1.22) on the intervals $I = [a, b)$, where $-\infty < a < b \le \infty$.

Theorem 3.1.10. *[26] Let* $f, g, h : I \to \mathbb{R}$ *be continuous functions and let for a positive constant* $0 < N < 1$,
$|f(t)| + |g(t)| \ge N$ *for all* $t \in I$. *Assume that* $\psi : I \to [o, \infty)$ *is an integrable function with property that there exists* $P \in (0, 1)$ *such that*

$$\int_a^t (1 + |f(t_1)| + |g(t_1)|)\psi(t_1)dt_1 \le P\psi(t) \tag{3.1.23}$$

for all $t \in I$. *If a function* $y \in C^2(I, \mathbb{R})$ *verifies the relation*

$$|\ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t)| \le \psi(t) \tag{3.1.24}$$

for all $t \in I$, *then there exists a unique solution* $s_1 : I \to \mathbb{R}$ *of equation (3.1.22), which verifies the following relation*

$$|y(t) - s_1(t)| \le \frac{P}{N(1-P)}\psi(t) \tag{3.1.25}$$

for all $t \in I$, *and* $s_1(a) = y(a)$.

Proof. Let $z_1(t) = y(t), z_2(t) = \dot{z}_1(t),\ \dot{z}_2(t) = -f(t)z_2(t) - g(t)z_1(t)$.
Let $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, A(t) = \begin{bmatrix} 0 & -1 \\ g(t) & f(t) \end{bmatrix}, H(t) = \begin{bmatrix} 0 \\ h(t) \end{bmatrix}$, then equation(3.1.22) transfer into

$$\dot{z}(t) + A(t)z(t) + H(t) = 0 \tag{3.1.26}$$

for all $t \in I$. By hypotheses ,since $|f(t)| + |g(t)| \geq N$, then
$\|A(t)\| = 1 + |f(t)| + |g(t)| \geq N$ and by relation (3.1.23) we obtain
$\int_a^t \|A(t)\| \, \psi(t_1)dt_1 \leq P\psi(t)$.
Now, let a function y satisfy the relation (3.1.24), since
$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$ and

$$\begin{aligned} \dot{z}(t) + A(t)z(t) + H(t) &= \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ g(t) & f(t) \end{bmatrix} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ h(t) \end{bmatrix} \\ &= \begin{bmatrix} \dot{y}(t) - \dot{y}(t) \\ \ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t) \end{bmatrix} \end{aligned}$$

Therefore,
$\|\dot{z}(t) + A(t)z(t) + H(t)\| = |\ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t)| \leq \psi(t)$. Hence, by Theorem (3.1.6), there exists solution such $s(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix}$ of equation (3.1.26) and $s(a) = z(a)$ such that $\|z(t) - s(t)\| \leq \frac{P}{N(1-P)}\psi(t)$ for all $t \in I$.
Therefore, there exists $s_1(t)$ satisfying equation (3.1.22) and $s_1(a) = y(a)$ such that
$|y(t) - s_1(t)| \leq \frac{P}{N(1-P)}\psi(t)$ for all $t \in I$. □
In the same manner and by Theorem (3.1.7) we can prove the Hyers-Ulam-Rassias stability for the equation (3.1.22) on the interval $J = (b, a]$, where $-\infty \leq b < a < \infty$.

Theorem 3.1.11. *[26] Let $f, g, h : J \to \mathbb{R}$ be continuous functions and let for a positive constant $0 < N < 1$,
$|f(t)| + |g(t)| \geq N$ for all $t \in J$. Assume that $\psi : J \to [o, \infty)$ is an integrable function with property that there exists $P \in (0, 1)$ such that*

$$\int_t^a (1 + |f(t_1)| + |g(t_1)|)\psi(t_1)dt_1 \leq P\psi(t) \tag{3.1.27}$$

for all $t \in J$. If a function $y \in C^2(J, \mathbb{R})$ verifies the relation

$$|\ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t)| \leq \psi(t) \tag{3.1.28}$$

for all $t \in J$. then there exists a unique solution $s_1 : J \to \mathbb{R}$ of equation (3.1.22), which verifies the following relation

$$|y(t) - s_1(t)| \leq \frac{P}{N(1-P)}\psi(t) \tag{3.1.29}$$

for all $t \in J$, and $s_1(a) = y(a)$.
Now in the same manner Corollary (3.1.8) and by Theorems (3.1.10) and (3.1.11), we obtain the following Corollary

Corollary 3.1.12. *[26] Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be continuous functions and let for a positive constant $0 < N < 1$, $|f(t)| + |g(t)| \geq N$ for all $t \in \mathbb{R}$. Assume that $\psi : \mathbb{R} \to [o, \infty)$ is an integrable function with property that there exists $P \in (0, 1)$ such that*

$$\left| \int_0^t (1 + |f(t_1)| + |g(t_1)|)\psi(t_1)dt_1 \right| \leq P\psi(t) \tag{3.1.30}$$

for all t in $\mathbb{R}$. If a function $y \in C^2(\mathbb{R}, \mathbb{R})$ verifies the relation

$$|\ddot{y}(t) + f(t)\dot{y}(t) + g(t)y(t) + h(t)| \leq \psi(t) \tag{3.1.31}$$

for all $t \in \mathbb{R}$, then there exists a unique solution $s_1 : \mathbb{R} \to \mathbb{R}$ of equation (3.1.22), which verifies the following relation

$$|y(t) - s_1(t)| \leq \frac{P}{N(1-P)}\psi(t) \tag{3.1.32}$$

for all $t \in \mathbb{R}$, and $s_1(0) = y(0)$.

Remark 3.1.13. *[26] The results can be applied to all differential equations of higher order by transferring it to system of first order.*

3.2 Hyers-Ulam Stability of Nonlinear System of Differential Equations

In 1961, the notion of practical stability was discussed in the monograph by Lasalle and Lefschetz [17]. In which they point out that stability investigations may not assure practical stability and vice versa. For example an aircraft may oscillate around a mathematically unstable path, yet its performance may be acceptable. Motivated by this, Weiss and Infante introduced the concept of finite time stability[20]. There are many studies about the relation between types of stability, Lyapunov stability and practical stability (see [17],[31],[2]). With these results in mind, in this section, we give a sufficient condition in order that the first order nonlinear system of differential equations has Hyers-Ulam stability and Hyers-Ulam-Rassias stability. In addition, we present the relation between practical stability and Hyers-Ulam stability and also Hyers-Ulam-Rassias stability.

3.2.1 Preliminaries and Auxiliary Results

Let $(\mathbb{B}, \|.\|)$ be a Banach space (real or complex), and let $J = [t_0, t_0 + T)$ for some $T > 0$, $t_0 \geq 0$. We consider two systems: a system

$$\dot{x} = f(t, x) \ , \forall t \in J, \tag{3.2.1}$$

where f is defined and continuous on $J \times \mathbb{B}$. The equilibrium state is at the origin : $f(t, 0) = 0, \forall t \in J$.
A system that depends on parameter $\epsilon \in (0, \epsilon_0]$,($\epsilon_0 \in (0, \infty)$) which is said to be perturbed system

$$\dot{x} = f(t, x) + p(t, x). \tag{3.2.2}$$

Let P be the set of all perturbations p satisfying $\|p(t, x)\| = \epsilon \leq \epsilon_0$ for all $t \in J$ and all x, let Q be a closed and bounded set of $\mathbb{B}$ containing the origin and let Q_0 be a subset of Q .

Definition 3.2.1. *Practical stability[17]*
Let $x^*(t, x_0, t_0)$ be the solution of (3.2.2) satisfying $x^*(t_0, x_0, t_0) = x_0$. If for each $p \in P$,i.e. $\epsilon \in (0, \epsilon_0]$, x_0 in Q_0 and each $t_0 \geq 0$, $x^*(t, x_0, t_0)$ in Q for all $t \in J$, then the origin is said to be (Q_0, Q, ϵ_0)-practically stable .
The slutions which start initially in Q_0 remain thereafter in Q .

Definition 3.2.2. *[39],[41] Let ϵ be a positive real number. We consider the system* (3.2.1) *with following differential inequality*

$$\|\dot{y}(t) - f(t, y(t))\| \leq \epsilon \ , \forall t \in J. \tag{3.2.3}$$

The equation (3.2.1) is generalized Hyers-Ulam stability (GHUs) if for each $\epsilon \in (0, \epsilon_0]$ and for each solution $y(t, t_0, x_0) \in C^1(J, \mathbb{B})$ of (3.2.3) there exists a solution $x \in C^1(J, \mathbb{B})$ of (3.2.1) with

$$|y(t) - x(t)| \leq K(\epsilon),$$

where $K(\epsilon)$ *is an expression of* ϵ *with* $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$ *.*

Definition 3.2.3. *[7],[44]*
We consider the system (3.2.1) *with following differential inequality*

$$\|\dot{y}(t) - f(t, y(t))\| \leq \varphi(t) \ \forall t \in J, \tag{3.2.4}$$

where $\varphi : J \to [0, \infty)$ *is a continuous function. The equation (3.2.1) is generalized Hyers-Ulam-Rassias stability (GHURs) with respect to* φ *if there exists* $K > 0$ *such that for each solution* $y(t, t_0, x_0) \in C^1(J, \mathbb{B})$ *of (3.2.4) there exists a solution* $x \in C^1(J, \mathbb{B})$ *of (3.2.1) with*

$$|y(t) - x(t)| \leq K\varphi(t), \forall t \in J$$

Definition 3.2.4. *[31] We say that* $V : J \times \mathbb{B} \to \mathbb{R}$ *is a Lyapunov function if* $V(t, x)$ *is continuous in* (t, x)*, bounded on bounded subset of* $\mathbb{B}$*.*

3.2.2 Hyers-Ulam Stability of System of Differential Equations

Lemma 3.2.5. *[27] Consider the following differential equation*

$$\dot{x} = f(t, x(t)) \ , t \in J \tag{3.2.5}$$

with initial condition

$$x_0 = x(t_0) \in Q_0, \tag{3.2.6}$$

where f *is defined and continuous on* $J \times \mathbb{B}$*, and equilibrium state is at the origin:* $f(t, 0) = 0 \ , \forall t \in J$*. The system (*3.2.5*),(*3.2.6*) to be (*Q_0, Q, ϵ_0*)-practically stable it is sufficient that there exists a continuous non increasing on the system (*3.2.5*) solutions Lyapunov function* $V(t, x)$ *such that*

$$\{x \in \mathbb{B} : V(t, x) \leq 1\} \subseteq Q \ , t \in J \tag{3.2.7}$$

$$Q_0 \subseteq \{x \in \mathbb{B} : V(t_0, x) \leq 1\} \tag{3.2.8}$$

Proof. We will prove by contradiction. Suppose that conditions (3.2.7) ,(3.2.8) are satisfied but there are $\tau \in J$ and $x_0 \in Q_0$ such that the solution $x(t) = x(t, x_0, t_0)$ of (3.2.5) leaves the set Q. From (3.2.7) follows inequality $V(\tau, x(\tau)) > 1$which contradicts the condition (3.2.8). Therefore our assumption is false and the equilibrium of system (3.2.5),(3.2.6) is (Q_0, Q, ϵ_0)-practically stable . □

Theorem 3.2.6. *[27] Consider two systems: the system of differential equation* (3.2.5)*,* (3.2.6) *and the system* (3.2.2) *. If equilibrium of* (3.2.5) *(at the origin) is* (Q_0, Q, ϵ_0)*-practically stable then the system* (3.2.5)*,* (3.2.6) *is generalized Hyers-Ulam stability.*

Proof. Since Q is closed and bounded set then there exists real number $M > 0$ such that $Q = \{x : \|x\| \leq M\}$.
Now, let $x^* = f(t, x_0, t_0)$ satisfying (3.2.3) for arbitrary $\epsilon \in (0, \epsilon_0]$, then x^* satisfies (3.2.2). Since the equilibrium of (3.2.5) is (Q_0, Q, ϵ_0)-practically stable then x^* in Q, that means that $\|x^*\| \leq M$. Since $M > 0$, $\epsilon > 0$ then there exists $s > 0$ such that $M = s\epsilon$.
Hence, $\|x^*\| \leq s\epsilon$ for all $t \in J$,$\lim_{\varepsilon \to 0} K(\varepsilon) = \lim_{\varepsilon \to 0} s\varepsilon = 0$. Obviously ,$w(t) = 0$ satisfies the equation (3.2.5) and

$$\|x^*(t) - w(t)\| \leq s\epsilon \ , \forall t \in J.$$

Hence, the equation (3.2.5) with initial condition (3.2.6) has generalized Hyers-Ulam stability. □

Corollary 3.2.7. *[27] For the system (*3.2.5*), (*3.2.6*) to be generalized Hyers-Ulam stability it sufficient that there exists a continuous non increasing on the system (*3.2.5*) solutions Lyapunov function* $V(t, x)$ *such that satisfies the conditions (3.2.7) and (3.2.8).*

Proof. Suppose that conditions (3.2.7), (3.2.8) are satisfied, then by lemma 3.2.5 the system (3.2.5),(3.2.6) is (Q_0, Q, ϵ_0)-practically stable. Hence, by theorem 3.2.6 the system has generalized Hyers-Ulam stability. □

Theorem 3.2.8. *[27] Consider the following differential equation*

$$\dot{x} = f(t, x(t)) \ , t \in J \tag{3.2.9}$$

with initial condition

$$x_0 = x(t_0) \in Q_0, \tag{3.2.10}$$

where f *is defined and continuous on* $J \times \mathbb{B}$*, and equilibrium state is at the origin :*$f(t, 0) = 0 \ , \forall t \in J.$
If equilibrium is (Q_0, Q, ϵ_0)*-practically stable and there exists* $\epsilon_1 > 0$ *such that* $\epsilon_1 \leq \varphi(t) \leq \epsilon \ \forall t \in J$ *then the system* (3.2.9)*,* (3.2.10) *is GHURs with respect to* φ*.*

Proof. Since Q is closed and bounded set then there exists real number $M > 0$ such that $Q = \{x : \|x\| \leq M\}$.
Now, let $x^* = f(t, x_0, t_0)$ satisfying (3.2.9), since $\varphi(t) \leq \epsilon$ then x^* satisfies (3.2.2). Since the equilibrium of (3.2.9) is (Q_0, Q, ϵ_0)-practically stable then x^* in Q, that mean that $\|x^*\| \leq M$. Since $M > 0$, $\epsilon_1 > 0$ then there exists $K > 0$ such that $M = K\epsilon_1$.
Then, $\|x^*\| \leq K\epsilon_1$ for all $t \in J$,hence $\|x^*\| \leq K\varphi(t)$ for all $t \in J$. Obviously, $w(t) = 0$ satisfies the equation (3.2.9) and

$$\|x^*(t) - w(t)\| \leq K\varphi(t) \ , \forall t \in J.$$

Hence, the equation (3.2.9) with initial condition (3.2.10) has generalized Hyers-Ulam-Rassias stability. □

Corollary 3.2.9. *[27] For the system (*3.2.9*), (*3.2.10*) to be generalized Hyers-Ulam stability it sufficient that there exsist a continuous nonincreasing on the system (*3.2.9*) solutions Lyapunov function $V(t, x)$ such that satisfies the conditions (3.2.7) and (3.2.8).*

Proof. Suppose that conditions (3.2.7), (3.2.8) are satisfied, then by lemma 3.2.5 the system (3.2.9), (3.2.10) is (Q_0, Q, ϵ_0)-practically stable. Hence, by theorem 3.2.8 the system has generalized Hyers-Ulam-Rassias stability. □

Theorem 3.2.10. *[27] Let $(\mathbb{B}, \|.\|)$ be a Banach space (real or complex), and let $J = [t_0, t_0 + T)$ for some $T > 0$, $t_0 \geq 0$. Consider two systems : a system*

$$\dot{x} = f(t, x) \ , \forall t \in J, \tag{3.2.11}$$

with initial condition

$$x(t_0) = 0 \in Q_0, \tag{3.2.12}$$

for a set Q_0, where f is defined, continuous on $J \times \mathbb{B}$ and satisfies Lipschitz condition. The equilibrium state is at the origin : $f(t, 0) = 0, \forall t \in J$.
A system that depends on parameter $\epsilon \in (0, \epsilon_0]$, $(\epsilon_0 \in (0, \infty))$ which is said to be perturbed system

$$\dot{x} = f(t, x) + p(t, x). \tag{3.2.13}$$

Let P be the set of all perturbations p satisfying $\|p(t, x)\| = \epsilon \leq \epsilon_0$ for all $t \in J$ and all x .

If the system of differential equation (3.2.11), (3.2.12) *has Hyers-Ulam stability with Hyers-Ulam constant K then the origin is (Q_0, Q, ϵ_0)-practically stable, where $Q = \{x : \|x\| \leq K\epsilon_0\}$, contains the origin.*

Proof. Let $\epsilon > 0, \epsilon \in (0, \epsilon_0]$ and let $x^* = f(t, x_0, t_0)$, $x_0 \in Q_0$ be a solution of (3.2.13), i.e. $\left\|\dot{x}^* - f(t, x^*)\right\| \leq \epsilon$. Since the system (3.2.11), (3.2.12) has Hyers-Ulam stability with constant $K > 0$ then there exists y a solution of (3.2.11), (3.2.12) with $\left\|x^* - y\right\| \leq K\epsilon$. By uniqueness of solution then y=0. Hence $\left\|x^*\right\| \leq K\epsilon \leq K\epsilon_0$. Thus the equilibrium of (3.2.11), (3.2.12) is (Q_0, Q, ϵ_0)-practically stable. □

Remark 3.2.11. *[20] In case $Q_0 \subset Q$ then we have expansive stability. If $Q \subset Q_0$ then we have contractive stability.*

Remark 3.2.12. *If we have a differential equation of n-order in a Banach space $\mathbb{B}_1$ then we reduce it to a differential equation of first order in Banach space $\mathbb{B} = \mathbb{B}_1^n$.*

List of Papers

Published papers

(1) Qusuay.H. Alqifiary, S.M. Jung, *On the Hyers-Ulam stability of differential equations of second order*, *Hindawi Publ. Corp. J. Abstract and Applied Analysis* , Article ID 483707(2014),8 pages.

(2) Qusuay.H. Alqifiary, S.M. Jung, *Hyers-Ulam stability of second-order linear dierential equations with boundary conditions*, *SYLWAN.* , 158(5),(2014),289-301 pages.

(3) Qusuay.H. Alqifiary, S.M. Jung, *Laplace transform and generalized Hyers-Ulam stability of linear differential equations*, Electronic Journal of Differential Equations. **2014** (2014), no. 80, 1-11.

(4) Jinghao Huang,Qusuay.H. Alqifiary, Yongjin Li, *Superstability of differential equations with boundary conditions*, Electronic Journal of Differential Equations. **2014** (2014), no. 215, 1-8.

(5) Qusuay.H. Alqifiary, Some properties of second order differential equations, Mathematica Moravica, Vol. 17-1 (2013), pp. 89-94.

(6) Qusuay.H. Alqifiary, *Note on the stability for linear systems of differential equations*, International Journal of Applied Mathematical Research, 3 (1) (2014) pp.15-22.

(7) Qusuay.H. Alqifiary, J. Kneevi-Miljanovi, *Note on the stability of system of differential equations x' = f(t; x(t))*, Gen. Math. Notes.,Vol. 20, No. 1, January 2014, pp. 27-33.

Accepted papers

(1) Jinghao Huang,Qusuay.H. Alqifiary, Yongjin Li, *On the generalized superstability of nth-order linear differential equations with initial conditions*, Publications de l'Institut Mathematique.

(2) Qusuay.H. Alqifiary, *On Hyers-Ulam-Rassias stability of linear differential equations with boundary conditions*, Southeast Asian Bulletin of Mathematics.

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